LOCAL EXISTENCE OF CLASSICAL SOLUTIONS TO SHALLOW WATER EQUATIONS WITH CAUCHY DATA CONTAINING VACUUM*

BEN DUAN†, ZHEN LUO‡, AND YUXI ZHENG§

Abstract. In this paper, we investigate the Cauchy problem for the rotating shallow water equations with physical viscosity. We obtain the local existence of classical solutions without assuming the initial height is small or a small perturbation of some constant status. Moreover, the initial vacuum is allowed and the spatial measure of the set of vacuum can be arbitrarily large. In particular, the initial height can even have compact support; in this case, a blow-up example is given.

Key words. viscous compressible rotating shallow water system, Cauchy problem, classical solution, blow-up

AMS subject classifications. 35Q35, 35M10, 76N10, 75U05

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1. Introduction. The nonlinear shallow water equations can be used to describe the horizontal structure of the atmosphere. They simulate the evolution of an incompressible fluid in response to gravitational and rotational accelerations. In general, it is modeled by the three-dimensional (3D) incompressible Navier–Stokes–Coriolis system in a rotating subdomain of \( \mathbb{R}^3 \) together with a nonlinear free moving surface boundary condition for which the stress tension is evolved at the air-fluid interface from above and the Navier boundary condition holds at the bottom. It is also regarded as an important extension of the two-dimensional (2D) compressible Navier–Stokes equations with additional rotating force, and the solutions present many types of motion. Usually, the nonlinear shallow water equations take the form

\[
\begin{align*}
ht + \text{div}(hu) &= 0, \\
(hu)_t + \text{div}(hu \otimes u) + gh\nabla h + f(hu)^\perp &= \mu \Delta u + (\mu + \lambda)\nabla(\text{div} u),
\end{align*}
\]

where \( x \in \Omega \subset \mathbb{R}^2, t \in \mathbb{R}^+, h(x,t) \) is the height of the fluid surface, \( u(x,t) \) is the horizontal velocity field, \( g > 0 \) is the gravity constant, \( f > 0 \) is the Coriolis frequency, and \( \mu \) and \( \lambda \) are the dynamical viscosities satisfying

\[
\mu > 0, \quad \mu + \lambda \geq 0.
\]

Utilizing scaling in \( (t, x, h) \), we can assume without loss of generality that \( g = 1 \) and

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\( f = 1 \). And for sophistication of the model, we extend the term \( h \nabla h \) to \( \nabla h^\gamma \), i.e., we shall consider (1.3):

\[
\begin{aligned}
&h_t + \text{div}(hu) = 0, \\
&(hu)_t + \text{div}(hu \otimes u) + \nabla h^\gamma + hu^\perp = \mu \Delta u + (\mu + \lambda)\nabla (\text{div} u),
\end{aligned}
\]

where \( \gamma > 1 \) is an arbitrary constant. Let \( \tilde{h} \) be a fixed nonnegative constant. We look for solutions \((h(x,t), u(x,t))\) to the Cauchy problem for (1.1) with the far field behavior

\[
u(x,t) \rightarrow 0, h(x,t) \rightarrow \tilde{h} \geq 0 \text{ as } |x| \rightarrow \infty
\]

and initial data

\[
(h, u)|_{t=0} = (h_0 \geq 0, u_0), \quad x \in \mathbb{R}^2,
\]

satisfying the compatibility condition

\[
-\mu \Delta u_0 - (\mu + \lambda)\nabla \text{div} u_0 + \nabla h_0^2 + h_0 u_0^\perp = h_0 g
\]

for some \( g \in D^1 \) with \( h_0^{1/2} g \in L^2 \), where \( h_0 \) and \( u_0 \) are functions. First, let us introduce the notation and conventions used throughout this paper. Integral domain may be omitted if it is the entire domain \( \mathbb{R}^2 \). For \( 1 < r < \infty \), we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

\[
\begin{aligned}
L^r = L^r(\mathbb{R}^2), & \quad D^{k,r} = \{ u \in L^1_{\text{loc}}(\mathbb{R}^2) \mid ||\nabla^k u||_{L^r} < \infty \}, \quad ||u||_{D^{k,r}} := ||\nabla^k u||_{L^r}, \\
W^{k,r} = L^r \cap D^{k,r}, & \quad H^k = W^{k,2}, \quad D^k = D^{k,2}, \quad D_0^k = \{ u \in D^k ; |u|_{\partial \Omega} = 0 \}.
\end{aligned}
\]

Moreover, the material derivative is defined as

\[
\dot{f} := f_t + u \cdot \nabla f.
\]

There is considerable literature studying fluid dynamics. The multidimensional Navier–Stokes system was investigated by Matsumura and Nishida [30, 31, 32], who proved global existence of smooth solutions for data close to a nonvacuum equilibrium, and later by Hoff for discontinuous initial data [21, 22]. Kazhikhov and Vaĭgant [25] obtained global existence of classical solutions in dimension \( N = 2 \) for special viscous coefficients with large initial data but for initial density away from zero. See also Desjardins and Lin [15], Mucha and Zajączkowski [33] and Itoh, Tanaka, and Tani [23]. For the case that the initial density is allowed to vanish, in fundamental work [28], Lions developed an existence theory of global in time weak solutions (see also Feireisl [16]). Better regularity of spatially periodic weak solutions in both two and three spatial dimensions was proved by Desjardins [13, 14] for small time. Later, for initial data near equilibrium, Danchin [11] and Danchin and Desjardins [12] found the optimal global well-posedness of the Cauchy problem in a functional space invariant by the natural scaling of the associated equations. Recently, for dimension \( N \geq 3 \), the local well-posedness of classical solutions containing vacuum were obtained by Cho, Choe, and Kim [1] and Cho and Kim [2, 6]. Such solutions were later shown to exist globally in time by Huang, Li, and Xin [20]. However, the existence of strong or classical solutions to the 2D Cauchy problem is still open.

For the shallow water system, there is a great deal of work. The local existence and uniqueness for classical solutions to the Cauchy–Dirichlet problem was studied in [37]...
using Lagrangian coordinates and Hölder space estimates with initial data in $C^{2+\alpha}$. Later, assuming the initial data is a small perturbation of a positive constant, Kloeden [24] and Sundbye [35, 36] proved global existence and uniqueness of classical solutions to the Cauchy–Dirichlet problem and also the Cauchy problem, using Sobolev space estimates and the energy method of Matsumura and Nishida [30, 31, 32]. Wang and Xu [38] obtained local solutions for any initial data and global solutions for small initial data, $h_0 - \bar{h}_0, u_0 \in H^{2+\epsilon}(\mathbb{R}^2)$ with $\bar{h}_0$ and $s > 0$. The result was improved by Chen, Miao, and Zhang in [5], who proved the local existence in time for general initial data and global existence in time for small initial data where $h_0 - \bar{h}_0 \in B^{0}_{2,1} \cap B^{1}_{2,1}$ and $u_0 \in B^{0}_{2,1}$ with the additional condition that $h \geq \bar{h}_0 > 0$. A global existence of strong solution in the space of Besov type was obtained recently by Hao, Hisao, and Li [19] for the shallow water equations involving the capillary term $h\nabla \Delta h$ with the initial data close to a positive constant equilibrium state. For arbitrarily large initial data, Bresch and Desjardins [7, 8] and Bresch, Desjardings, and Lin [9] proved the global existence of weak solutions for 2D shallow water equations in bounded domain with periodic boundary conditions, where the friction term and the capillary term are involved. Later, Li, Li, and Xin showed the global existence of weak entropy solution for the initial boundary value problem to one-dimensional compressible flows [27]. The same result was obtained by Guo, Jiu, and Xin [17] for the case of multidimensional spherically symmetric weak solutions.

We remark that the 2D problem is the critical case for the standard Sobolev embedding theorems in unbounded domain and one cannot bound the $L^p$-norm of the velocity just in terms of the $L^2$-norm of the gradient of it. As a consequence, in unbounded domains, a priori estimates in some existence theory in 3D, such as [2], cannot be applied for the 2D case. So far, we only know results in 2D bounded domains [7, 8, 9]. In fact, the existence of strong or classical solutions to the Cauchy problem for (1.3)–(1.5) is open, especially, when the initial height has compact support. In this paper, basic energy estimates are derived on the material derivatives of the velocity, $\dot{u} = \partial_t u + (\nabla \cdot u) u$, in the spirit of D. Hoff’s work in [21]. Actually, one of the main difficulties in this paper is to derive $L^p$-bounds on the velocity $u$ and the material derivatives of the velocity $\dot{u}$: combining some substantial estimates on a suitably spatial weighted norm of $\nabla u$ and $\nabla \dot{u}$ with the Caffarelli–Kohn–Nirenberg inequality yields the $L^p$-norm of $u$ and $\dot{u}$ for some constant $p = p(\mu; \lambda)$.

In [10], several open problems about shallow water equations are mentioned. Particularly, in section 4, the authors remark on strong solution that “nothing has been done so far, to obtain better results such as local existence of strong solution with initial data including vacuum. Remark that such a situation is important from a physical point of view: the dam break situation”; the dam break problem is the case in which initial height has compact support. In fact, the possible appearance of vacuum is one of the major difficulties when trying to prove existence and strong regularity results to the hyperbolic-parabolic system (1.1), which possesses strong degeneracies near vacuum and singularities in the vacuum region. See [7, 8, 9] for weak solutions containing vacuum and [24, 35, 36, 38, 5] for higher regularity solutions away from vacuum, assuming that the initial height is a small perturbation of a positive constant. In this paper, we investigate the existence of classical solutions for nonnegative initial height, and the set of initial vacuum can be arbitrary large. Therefore, we are able to deal with the problem without assuming the initial height is close to a nonvacuum equilibrium. Furthermore, the initial far fields can be vacuum or nonvacuum. Note that due to the strong nonlinearity of system (1.1), the problem of existence of solutions for large initial data is difficult and previous results for classical solutions
only consider the case of small initial data; see [24, 35, 36]. Our existence theory of classical solutions in this paper has no restriction on smallness of initial data.

We remark that for a 2D Navier–Stokes system in bounded domain, results obtained by Desjardins in [13] showed that the maximal norm of the density controls the breakdown of weak solution, even if vacuum forms in the fluid. That is, vacuum does not yield additional singularities in this case. In this paper, for the Cauchy problem of rotating shallow water equations with spherically symmetry initial data, we obtain that the local smooth solution \((h, u) \in C^1([0; T]; H^s)(s > 2)\) has to blow up in finite time if the initial height has compact support. Therefore, the solutions in our main result, Theorem 3.9, give a kind of suitable class of solutions which are expected to exist globally in time in the case \(\tilde{h} = 0\).

As above, the main technical difficulties in this papers are to deal with the initial height allowing vacuum, with no restriction on the smallness of initial data or perturbation in the sense of \(||h_0 - \hat{h}_0||\), and to gain high regularity estimates to show the solution is classical. We obtain the local existence of classical solutions in this paper; moreover, we give a blow-up example when the far field state is \(\tilde{h} = 0\).

The structure of the paper is as follows. In the next section, we recall some useful well-known lemmas. In section 3 we prove the local existence of classical solutions to the Cauchy problem for all sizes of the initial data, with or without initial vacuum. In section 4, assuming the initial height has compact support, we construct a radially symmetric solution and we prove that this solution has to blow up in finite time.

2. Preliminaries. First, the following Caffarelli–Kohn–Nirenberg inequality will play a key role in obtaining the \(L^p\) estimate of \(u\) when the constant far field state \(h = 0\).

**Lemma 2.1 (see [3, 4]).** For \(\alpha \in (0, 2)\), the following estimates hold for all \(u \in C_0^\infty(\mathbb{R}^2)\):

\[
\int |x|^{\alpha - 2}|u|^2dx \leq \frac{\alpha^2}{4} \int |x|^{\alpha}|\nabla u|^2dx, \quad \|u\|_{L^{4/\alpha}}^2 \leq C(\alpha) \int |x|^{\alpha}|\nabla u|^2dx.
\]

The following Lions–Aubin lemma will be used later.

**Lemma 2.2 (see [34]).** Let \(X_0, X,\) and \(X_1\) be three Banach spaces with \(X_0 \subset X \subset X_1\). Suppose that \(X_0\) is compactly embedded in \(X\) and that \(X\) is continuously embedded in \(X_1\).

(i) Let \(G\) be bounded in \(L^p(0, T; X_0)\), where \(1 \leq p < \infty\), and let \(\partial G/\partial t\) be bounded in \(L^1(0, T; X_1)\). Then \(G\) is relatively compact in \(L^p(0, T; X)\).

(ii) Let \(F\) be bounded in \(L^\infty(0, T; X_0)\) and \(\partial F/\partial t\) be bounded in \(L^r(0, T; X_1)\), where \(r > 1\). Then \(F\) is relatively compact in \(C([0, T]; X)\).

The following Schauder fixed point theorem will be needed to obtain the local existence of strong solutions in a bounded smooth domain.

**Lemma 2.3 (see [18]).** Let \(\mathcal{B}\) be a compact convex set in a Banach space \(\mathcal{B}\) and let \(T\) be a continuous mapping of \(\mathcal{B}\) into itself. Then \(T\) has a fixed point, that is, \(T x = x\) for some \(x \in \mathcal{B}\).

Next, the well-known Gagliardo–Nirenberg inequality will be used frequently (see [26]).

**Lemma 2.4 (Gagliardo–Nirenberg).** For \(p \geq 2, q \in (1, \infty),\) and \(r \in (2, \infty),\) there exists some generic constant \(C > 0\) which may depend on \(q, r\) such that for \(f \in H^1(\mathbb{R}^2)\) and \(g \in L^q(\mathbb{R}^2) \cap D^{1,r}(\mathbb{R}^2)\), we have

\[
\|f\|_{L^p} \leq C\|f\|_{L^2}^{2/p} \|\nabla f\|_{L^2}^{(p-2)/p},
\]
\( \|g\|_{C^2(\mathbb{R}^2)} \leq C \|g\|_{L^q}^{2(r-2)/(2r+q(r-2))} \|\nabla g\|_{L^r}^{2r/(2r+q(r-2))}. \)

It is quite well known that for bounded smooth domain \( \Omega \subset \mathbb{R}^2 \), the linear hyperbolic problem
\[
\begin{cases}
\partial_t h + \text{div}(hv) = 0 & \text{in } \Omega \times (0, T), \\
h(x, 0) = h_0 & \text{in } \Omega,
\end{cases}
\]
where \( v \) is a known vector field in \( \Omega \times (0, T) \) such that
\[
v \in C([0, T]; H_0^1 \cap H^3) \cap L^2(0, T; H^4),
\]
has a unique strong solution \( h \) for any regular initial data \( h_0 \) because the vector field \( v \) is sufficiently smooth (see [6]). So we state the result without proof.

**Lemma 2.5.** Assume that (2.5) holds and that \( \tilde{h}_0 \geq 0, \tilde{h}_0 - h_0 \in H^3(\Omega) \). Then
\[
\begin{align*}
&\text{(i) there exists a unique solution } h
\text{ to the problem (2.4) such that } h - \tilde{h} \in C([0, T]; H^3), \quad h_t \in C([0, T]; H^2);
\end{align*}
\]
\[
\text{(ii) the solution } h \text{ satisfies the estimate }
\|h(t) - \tilde{h}\|_{H^3} \leq (\|h_0 - \tilde{h}\|_{H^3} + \tilde{h}) \exp \left( C \int_0^t \|v(s)\|_{D^{1,1}} ds \right)
\]
for some universal constant \( C \);
\[
\text{(iii) the solution } h \text{ is represented by the formula }
\]
\[
h(x, t) = h_0(U(0; x, t)) \exp \left\{ - \int_0^t \text{div } v(U(s; x, t), s) ds \right\},
\]
where \( U \in C([0, T]; [0, T] \times \Omega) \) is the solution to the initial value problem
\[
\begin{cases}
\frac{\partial}{\partial s} U(s; x, t) = v(s; U(s; x, t)), & 0 \leq s \leq T, \\
U(t; x, t) = x, & 0 \leq s \leq T, \quad x \in \Omega.
\end{cases}
\]

Next, we consider the linear parabolic problem
\[
\begin{cases}
h u_t + hv \cdot \nabla u - \mu \Delta u - (\mu + \lambda) \nabla (\text{div } u) + \nabla h^\gamma + h u^\perp = 0 & \text{in } (0, T) \times \Omega, \\
u = 0 & \text{on } (0, T) \times \partial \Omega, \\
u(x, 0) = u_0 & \text{in } \Omega,
\end{cases}
\]
where \( h \) is a known scalar field in \( (0, T) \times \Omega \) such that
\[
h, h^\gamma \in C([0, T]; H^3), \quad h_t, (h^\gamma)_t \in C([0, T]; H^2), \quad h \geq \delta \text{ on } [0, T] \times \Omega,
\]
for some constant \( \delta > 0 \). Recall that \( L := -\mu \Delta - (\mu + \lambda) \text{div} \) is a strongly elliptic operator (see [1], for instance). Then applying a standard method such as a semi-discrete Galerkin method or the method of continuity, we can prove the following
existence and regularity results on solutions to the linear parabolic problem (2.9). See also [6] for a similar result to Navier–Stokes equations.

**Lemma 2.6.** (i) Assume that $u_0 \in H^1_0$, (2.5), and (2.10) hold. Then there exists a unique strong solution $u$ to the problem (2.9) such that

$$u \in C([0, T]; H^1_0) \cap L^2(0, T; H^2), \quad u_t \in L^2(0, T; L^2).$$

(ii) If in addition $u_0 \in H^1_0 \cap H^2$ and $v_t \in L^\infty(0, T; L^2)$, then the solution $u$ satisfies

$$u \in L^\infty(0, T; H^2), \quad u_t \in L^2(0, T; H^1_0), \quad u_{tt} \in L^2(0, T; H^{-1}).$$

(iii) Finally, if $u_0 \in H^1_0 \cap H^3$, $u_0(0) = h(0)^{-1}(-\nabla h^2(0) - Lu_0) - u^1(0) - v(0) \cdot \nabla u(0) \in H^1_0$, and $v_t \in L^\infty(0, T; H^1_0)$, then the solution $u$ satisfies

$$u \in L^\infty(0, T; H^3), \quad u_t \in L^2(0, T; H^2), \quad u_{tt} \in L^2(0, T; L^2).$$

3. **Local existence of classical solutions.** In this section, for simplicity, we assume $\gamma = 2$ and all results are valid for other cases. We look for the local classical solutions, $(h(x, t), u(x, t))$, to the Cauchy problem for (1.3) with the far field behavior (1.4) and initial data (1.5).

3.1. **Uniform a priori estimates for the linearized problem.** In this subsection, we derive some uniform local (in time) a priori estimates for strong solutions $(h, u)$ to the linearized problem (2.4), (2.9), which are stated as in Lemmas 2.5 and 2.6. The estimates we obtained here are independent of the lower bound $\delta$ of $h_0$ and size of the domain $\Omega$. The main difficulty is how to obtain the $L^p$ norm for $u$ itself and the material derivatives of the velocity $\dot{u}$ for some $p \geq 2$.

The case of $\bar{h} > 0$ is easier because the following lemma holds.

**Lemma 3.1.** If $\bar{h} > 0$, then there exists some constant $C(\bar{h}) > 0$ such that the following estimate holds for $h - \bar{h} \in L^2$, $v \in D^1, h^{1/2}v \in L^2$:

$$\|\nabla v\|_{L^2}^2 \leq C \left( \int h|v|^2 dx + \|\bar{h} - h\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \right).$$

**Proof.** Equation (3.1) follows directly from Lemma 2.4 and the following simple fact:

$$\bar{h} \int |v|^2 dx = \int h|v|^2 dx + \int (\bar{h} - h)|v|^2 dx \leq \int h|v|^2 dx + C \|\bar{h} - h\|_{L^2} \|v\|_{L^2} \|\nabla v\|_{L^2}.$$ 

In the rest of this subsection, we assume $\bar{h} = 0$ and deduce some local in time a priori estimates for this case. Set

$$c_0 := 1 + \|h_0^{1/2}u_0\|_{L^2}^2 + \|(h_0, h_0^0)\|_{H^1}^2 + \|u_0\|_{D^{1/2}}^2$$

$$+ \|(h_0^2, |\nabla u_0|, h_0^{1/2}|g|, h_0^{1/2}u_0)(1 + |x|^{\alpha/2})\|_{L^2}^2 + \|g\|_{D^{1/2}}^2,$$

where

$$\alpha := \frac{\mu}{4(2\mu + \lambda)} \in (0, 1/8].$$
We define

\[
\Phi(v, t) := 1 + \sup_{0 \leq s \leq t} \left( \|\nabla v\|_{H^1}^2 + \|x^{\alpha/2}\nabla v\|_{L^2}^2 \right) + \int_0^t \|\nabla v\|_{H^1 \cap W^{1,4/\alpha}}^2 ds.
\]

From now on, several positive generic constants depending on \(\mu, \lambda\) are denoted by \(C\) and we always assume that \(0 \leq t \leq T \leq 1\). We start with the following energy estimate for \((h, u)\) and the preliminary \(L^2\) bounds for \(\nabla u\) and \(h^{1/2} \dot{u}\), in the spirit of Hoff’s work [21].

**Lemma 3.2.** Let \((h, u)\) be a smooth solution of (2.4), (2.9). Then

\[
\sup_{0 \leq s \leq t} \int (h|u|^2 + h^4) dx + \int_0^t \int |\nabla u|^2 dx dt \leq C_0 \exp \left\{ \Phi(v, t) t^{1/2} \right\}
\]

and

\[
\sup_{0 \leq s \leq t} \|\nabla u\|_{L^2}^2 + \int_0^t \int |h|\dot{u}|^2 dx dt \leq C_0^2 \exp \{C_0 t \exp \{C \Phi(v, t)\}\},
\]

where \(c_0\) as in (3.2) and \(\Phi(v, t)\) in (3.4).

**Proof.** Note that (2.4) implies that \(h^2\) satisfies

\[
h_t^2 + v \cdot \nabla h^2 + 2h^2 \text{div} v = 0.
\]

It is easy to check that

\[
\left(\|(h, h^2)\|_{L^2}^2, \right) \leq C \|
abla v\|_L^\infty \|(h, h^2)\|_{L^2}^2,
\]

which together with (2.7) yields that

\[
\sup_{0 \leq s \leq t} (\|(h, h^2)\|_{L^2}^2 + \|(h, h^2)\|_{L^\infty}^2)
\leq C_0 \exp \left\{ \int_0^t \|\nabla v\|_{L^\infty} ds \right\}
\leq C_0 \exp \left\{ \Phi(v, t) t^{1/2} \right\}.
\]

Now, the energy estimate gives

\[
\frac{1}{2} \left( \int h|u|^2 dx \right)_t + \int (\mu|\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2) dx \leq C_0 h^2 \|\nabla u\|_{L^2}^2 + \delta \|
abla u\|_{L^2}^2,
\]

which together with (3.9) yields (3.5).

Next, multiplying (2.9) by \(\dot{u}\), then integrating the resulting equality over \(\Omega\), leads to

\[
\int h|\dot{u}|^2 dx = \int (-\dot{u} \cdot \nabla h^2 + \mu \Delta u \cdot \dot{u} + (\lambda + \mu) \nabla \text{div} u \cdot \dot{u} - h u^1 \cdot \dot{u}) dx
\]

\[
\quad := \sum_{i=1}^4 M_i,
\]
Using (2.4) and integrating by parts give

\[ M_1 = - \int \hat{\mathbf{u}} \cdot \nabla h^2 dx \]
\[ = \int (\text{div } \mathbf{u}) h^2 - (\mathbf{v} \cdot \nabla \mathbf{u}) \cdot \nabla h^2) dx \]
\[ = \left( \int \text{div } \mathbf{u} h^2 dx \right)_t + \int (h^2 \text{div } \mathbf{u} + h^2 \partial_i \mathbf{v}^j \partial_j \mathbf{u}^i) dx \]
\[ \leq \left( \int \text{div } \mathbf{u} h^2 dx \right)_t + C \epsilon_0^2 \exp \left\{ C \Phi(v, t) t^{1/2} \right\} \left( \| \nabla \mathbf{u} \|_{L^2} + \| \nabla \mathbf{v} \|_{L^2} \right), \]
due to (3.9). Integration by parts implies

\[ M_2 = \int \mu \Delta \mathbf{u} \cdot \dot{\mathbf{u}} dx \]
\[ = - \frac{\mu}{2} (\| \nabla \mathbf{u} \|_{L^2}^2)_t - \mu \int \partial_i \mathbf{u}^j \partial_i (\mathbf{v}^k \partial_k \mathbf{u}^j) dx \]
\[ \leq - \frac{\mu}{2} (\| \nabla \mathbf{u} \|_{L^2}^2)_t + C \| \nabla \mathbf{v} \|_{L^\infty} \| \nabla \mathbf{u} \|_{L^2}^2. \]

Similarly,

\[ M_3 = - \frac{\lambda + \mu}{2} (\| \text{div } \mathbf{u} \|_{L^2}^2)_t - (\lambda + \mu) \int \text{div } \text{div } (\mathbf{v} \cdot \nabla \mathbf{u}) dx \]
\[ \leq - \frac{\lambda + \mu}{2} (\| \text{div } \mathbf{u} \|_{L^2}^2)_t + C \| \nabla \mathbf{v} \|_{L^\infty} \| \nabla \mathbf{u} \|_{L^2}^2 \]
and

\[ M_4 = - \int h \mathbf{u}^\perp \cdot \dot{\mathbf{u}} dx \leq \delta \int h|\dot{\mathbf{u}}|^2 dx + C \delta \int h|\mathbf{u}|^2 dx. \]

Combining (3.5), (3.8), and (3.11)–(3.15) with Gronwall’s inequality gives (3.6). □

The following two lemmas mainly deal with the estimates on the spatial weighted norm of \( \nabla \mathbf{u} \) and \( \nabla \dot{\mathbf{u}} \), which are needed to overcome the difficulties from a possibly large set of initial vacuum and failure of Sobolev embedding theorems in critical space \( \mathbb{R}^2 \).

**Lemma 3.3.** Let \((h, \mathbf{u})\) be as in Lemma 3.2. Then

\[ B'(t) + \int h|\dot{\mathbf{u}}|^2 |x|^{\alpha} dx \]
\[ \leq C \left( 1 + \| \nabla \mathbf{v} \|_{L^\infty} + \| \mathbf{v} \|_{L^\infty} \right) \left\| |x|^\alpha/2 \nabla \mathbf{u} \right\|_{L^2}^2 + 2 \mu \alpha^4 \left\| |x|^\alpha/2 \nabla \dot{\mathbf{u}} \right\|_{L^2}^2 \]
\[ + C \beta \epsilon_0^2 t \exp \left\{ C \Phi(v, t) \right\}, \]

where

\[ B(t) := \int |x|^{\alpha} \left( \frac{\mu}{2} |\nabla \mathbf{u}|^2 + \frac{\lambda + \mu}{2} |\text{div } \mathbf{u}|^2 - h^2 \text{div } \mathbf{u} + \beta h^4 + h|\mathbf{u}|^2 \right) dx \]
\[ \geq \frac{\mu}{4} \int |x|^{\alpha} \left( |\nabla \mathbf{u}|^2 + h^4 + h|\mathbf{u}|^2 \right) dx \]

for suitably large \( \beta(\mu, \lambda) > 0. \)
Proof. Multiplying (2.9) by \( \hat{u}|x|^\alpha \) and then integrating the resulting equality over \( \Omega \) leads to
\[
\int h|\hat{u}|^2 |x|^\alpha dx = \int |x|^\alpha (-\hat{u} \cdot \nabla h^2 + \mu \Delta u \cdot \hat{u} + (\lambda + \mu) \nabla \text{div} u \cdot \hat{u} - h\mathbf{u}^\perp \cdot \hat{u}) dx
\]
\[
:= \sum_{i=1}^{4} M_i'.
\]
Using (2.4) and integrating by parts yield
\[
M_1' = -\int |x|^\alpha \hat{u} \cdot \nabla h^2 dx
\]
\[
= \int ((|x|^\alpha \text{div} u)h^2 + \alpha x \cdot \hat{u}|x|^{-2} h^2 - |x|\alpha (v \cdot \nabla u) \cdot \nabla h^2) dx
\]
\[
= \left( \int |x|^\alpha \text{div} u h^2 dx \right)_t + \alpha \int h^2 |x|^{-2} x \cdot \hat{u} dx
\]
\[
+ \int h^2 |x|\alpha (\text{div} u v + \partial_i v_{j} \partial_j u_i) dx
\]
\[
- \alpha \int |x|\alpha h^2 h^2 : (\text{div} u - v \cdot \nabla u) dx
\]
\[
\leq \left( \int |x|^\alpha h^2 \text{div} u dx \right)_t + \alpha |x|\alpha \int |x|\alpha \nabla \hat{u}^2 dx + C \int h^4 |x|^\alpha dx
\]
(3.18)
\[
+ C|h^2||L^\infty||x|\alpha/2 \nabla v|^2_{L^2} + C|h^2||L^\infty||x|\alpha/2 \nabla u|^2_{L^2},
\]
where in the last two inequalities we have used (2.1). Integration by parts and using (2.1) imply
\[
M_2' = \mu \int |x|^\alpha \Delta u \cdot \hat{u} dx
\]
\[
= -\frac{\mu}{2} \left( \||x|\alpha/2 \nabla u|^2_{L^2} \right)_t - \mu \int |x|^\alpha \partial_i u_j \partial_j (v^i \partial_k u_j) dx
\]
\[
- \alpha \mu \int \partial_i u_j \hat{u}^j |x|\alpha - 2 x_i dx
\]
(3.19)
\[
\leq -\frac{\mu}{2} \left( \||x|\alpha/2 \nabla u|^2_{L^2} \right)_t + C (||\nabla v||_{L^\infty} + ||v||_{L^\infty}) ||\nabla u|| \cdot |x|^\alpha/2 ||^2_{L^2}
\]
\[
+ C \|\nabla u|| \cdot |x|^\alpha - 1 ||_{L^2} + C ||v||_{L^\infty} \||\nabla u||^2_{L^2}
\]
\[
\leq -\frac{\mu}{2} \left( \||x|\alpha/2 \nabla u|^2_{L^2} \right)_t + C (1 + ||\nabla v||_{L^\infty} + ||v||_{L^\infty}) |||x|\alpha/2 \nabla u||^2_{L^2}
\]
\[
+ \frac{\mu}{2} \alpha \|||x|\alpha/2 \nabla \hat{u}||^2_{L^2} + C \|v||_{L^\infty} \||\nabla u||^2_{L^2},
\]
and similarly,
(3.20)
\[
M_3' \leq -\frac{\lambda + \mu}{2} \left( \||x|\alpha/2 \text{div} u||^2_{L^2} \right)_t + C (1 + ||\nabla v||_{L^\infty} + ||v||_{L^\infty}) \|||x|\alpha/2 \nabla u||^2_{L^2}
\]
\[
+ \frac{\mu}{2} \alpha \|||x|\alpha/2 \nabla \hat{u}||^2_{L^2} + C \|v||_{L^\infty} \||\nabla u||^2_{L^2}.
\]
Next,

\[ M'_4 = - \int h \mathbf{u} \cdot \dot{\mathbf{u}} |x|^\alpha \, dx \]

(3.21)

\[ \leq C \delta \| \sqrt{n} |x|^\alpha \|_{L^2}^2 + \delta \int h |\dot{\mathbf{u}}|^2 |x|^\alpha \, dx. \]

Now, multiplying (2.9) by \( |x|^\alpha \) then integrating the resulting equation over \( \Omega \) and using (2.4) yield

\[
\left( \int \frac{1}{2} h |x|^\alpha \, dx \right)_{t}
= \int \left( \frac{\alpha}{2} |x|^\alpha \frac{h}{h} + x + h^2 \nabla \mathbf{u} |x|^\alpha + \alpha h^2 \mathbf{u} \cdot \mathbf{x} |x|^\alpha - 2 \right) \, dx
\]

(3.22)

\[ \leq \int (\mu |\nabla \mathbf{u}|^2 |x|^\alpha + \alpha \mu |x|^\alpha - 2 \mathbf{u} (\nabla \mathbf{u}) \mathbf{x}) \, dx \]

\[ - \int (\lambda + \mu) |\nabla \mathbf{u}|^2 |x|^\alpha + \alpha (\lambda + \mu) |x|^\alpha - 2 (\nabla \mathbf{u}) \mathbf{x} \, dx \]

\[ \leq \int h^4 |x|^\alpha \, dx + C( \| h \|_{L^\infty} \| \mathbf{v} \|_{L^\infty} + 1) \| \nabla \mathbf{u} \|_{L^2}^2. \]

Multiplying (3.7) by \( h^2 |x|^\alpha \), we obtain that

\[
\left( \int |x|^\alpha h^4 \, dx \right)_{t} \leq C \int |x|^\alpha h^4 |\mathbf{v}| \, dx + C \int |x|^\alpha h^4 |\nabla \mathbf{v}| \, dx \]

\[ \leq C \| h^2 \|_{L^\infty} \left( \int |x|^\alpha h^4 \, dx \right)^{1/2} \| |x|^\alpha \nabla \mathbf{v}| \|_{L^2} \]

(3.23)

\[ \leq C c_0^{1/2} \exp \{ C \Phi(v, t) \} \left( \int |x|^\alpha h^4 \, dx \right)^{1/2}. \]

Then

\[
\sup_{0 \leq t \leq T} \int |x|^\alpha h^4 \, dx \leq C c_0 t \exp \{ C \Phi(v, t) \}, \]

which together with (3.23) gives

(3.24)

\[ \left( \int |x|^\alpha h^4 \, dx \right)_{t} \leq C c_0^{2t} \exp \{ C \Phi(v, t) \}. \]

Combining (3.17)–(3.24) implies (3.16). \( \square \)

**Lemma 3.4.** Let \((h, \mathbf{u})\) be as in Lemma 3.2. Then the following estimate holds for \( q = 4/\alpha \):

(3.25)

\[
\sup_{0 \leq t \leq T} \left( \| \nabla h \|^2_{L^2 \cap L^q} + \| \nabla^2 \mathbf{u} \|^2_{L^2} + \int (1 + |x|^\alpha) \left( |\nabla \mathbf{u}|^2 + h^4 |\dot{\mathbf{u}}|^2 + h |\mathbf{u}|^2 \right) \, dx \right) \]

\[ + \int_0^t \int (1 + |x|^\alpha) |\nabla \dot{\mathbf{u}}|^2 \, dx \, dt + \int_0^t \| \nabla^2 \mathbf{u} \|^2_{L^q} \, dt \]

\[ \leq C c_0^{\exp \{ C \exp \{ C \Phi(v, t) \} t^{(q-2)/(2(q-1))} \}}. \]
Proof. First, let \( v^k \) be the \( k \)th component of a vector \( v \) and denote the operator \( \frac{\partial}{\partial x_k} \) by \( \partial_k \), \( k = 1, 2 \). Operating \( \partial_t + \partial_k(v^k \cdot \cdot \cdot) \) on each term of (2.9) gives

\[
\begin{align*}
\frac{\partial}{\partial x_k}\partial_t u + h v \cdot \nabla \dot{u} = & \quad \mu \Delta \dot{u} + (\mu + \lambda) \nabla \text{div} \dot{u} - \mu (\partial_k(\partial_t v^k \partial_t u) - \partial_t(\partial_k v^k \partial_t u) + \partial_t(\partial_j v^j \partial_k u)) \\
& - h(\dot{u})^1 + (\mu + \lambda) (\nabla (\text{div} v \text{div} u) - \nabla (\partial_j v^j \partial_k u)) - \nabla (\text{div} v^1 \text{div} u)) \\
& + \text{div} (h^2 \nabla v^1) + \nabla (h^2 \text{div} v).
\end{align*}
\]

Multiplying (3.26) by \( \dot{u} \) and integrating over \( \Omega \), we have by (3.6) and (3.5)

\[
\begin{align*}
\left( \int h|\dot{u}|^2 dx \right)_t + \mu \int |\nabla \dot{u}|^2 dx + (\lambda + \mu) \int |\text{div} \dot{u}|^2 dx \\
\leq C \int |\nabla v|^2 |\nabla u|^2 dx + \int h^2 |\nabla v|^2 dx \\
\leq C(|\nabla v|^{(q-2)/(q-1)} |\nabla^2 v|^{2/(q-1)} (\|\nabla u\|_{L^2}^2 + \|h\|_{L^2}^2) \\
\leq Cc_0^2 \exp \left\{ Cc_0 \exp \{ C\Phi(v, t) \} \right\} \|\nabla v\|_{L^2}^{(q-2)/(q-1)} \|\nabla^2 v\|_{L^q}^{(q-2)/(q-1)},
\end{align*}
\]

which yields

\[
\begin{align*}
\sup_{0 \leq \tau \leq t} \int h|\dot{u}|^2 dx + \mu \int_0^t \int |\nabla \dot{u}|^2 dx dt + (\lambda + \mu) \int_0^t \int |\text{div} \dot{u}|^2 dx dt \\
\leq Cc_0^2 \exp \left\{ Cc_0 \exp \{ C\Phi(v, t) \} \right\} \tau^{(q-2)/(2(q-1))},
\end{align*}
\]

where \( q = 4/\alpha \). Similarly, multiplying (3.26) by \( \dot{u} |x|^\alpha \) and integrating over \( \Omega \) lead to

\[
\begin{align*}
\left( \frac{1}{2} \int |x|^\alpha h|\dot{u}|^2 dx \right)_t + \mu \int |x|^\alpha |\nabla \dot{u}|^2 dx + (\lambda + \mu) \int |x|^\alpha |\text{div} \dot{u}|^2 dx \\
\leq \alpha \int h|v||\dot{u}|^2 |x|^{-1} dx + \mu \alpha \int |x|^{-1} |\nabla \dot{u}| |\dot{u}| dx + (\lambda + \mu) \alpha \int |x|^{-1} |\text{div} \dot{u}| |\dot{u}| dx \\
+ C \int (|\nabla u| |\nabla v| + h^2 |\nabla v|) (|\nabla \dot{u}| |x|^\alpha + \alpha |\dot{u}| |x|^{-1}) dx \\
\leq C\Phi^{1/2}(v, t) \int |x|^\alpha h|\dot{u}|^2 dx + \alpha^3 (2\mu + \lambda) \int |x|^\alpha |\nabla \dot{u}|^2 dx \\
+ \delta \int |x|^\alpha |\nabla \dot{u}|^2 dx + C_6 \| \nabla v \|_{L^\infty}^2 \int |x|^\alpha (|\nabla u|^2 + h^4) dx.
\end{align*}
\]

By (3.3), we have

\[
\begin{align*}
\int |x|^\alpha h|\dot{u}|^2 dx \quad + \mu \int |x|^\alpha |\nabla \dot{u}|^2 dx + (\lambda + \mu) \int |x|^\alpha |\text{div} \dot{u}|^2 dx \\
\leq C\Phi^{1/2}(v, t) \int |x|^\alpha h|\dot{u}|^2 dx + C\| \nabla v \|_{L^\infty}^2 B(t).
\end{align*}
\]
Therefore, we get for \( q = 4/\alpha \),

\[
\left( B(t) + \int |x|^\alpha h \hat{\mathbf{u}}^2 \, dx \right) + \frac{\mu}{2} \int |x|^\alpha |\nabla \hat{\mathbf{u}}|^2 \, dx \\
\leq C \Phi^{1/2}(v, t) \int |x|^\alpha h \hat{\mathbf{u}}^2 \, dx + Cc_0^2 \exp \left\{ C \Phi(v, t) t^{1/2} \right\} \Phi(v, t)
\]

\[
(3.28)
\]

(3.28) gives

\[
\left( B(t) + \int |x|^\alpha h \hat{\mathbf{u}}^2 \, dx \right) \leq C c_0 \exp \left\{ C \Phi(v, t) t^{1/2} \right\} \Phi(v, t),
\]

which together with (3.28) yields

\[
\sup_{0 \leq s \leq t} \left( B(t) + \int |x|^\alpha h \hat{\mathbf{u}}^2 \, dx \right) \leq C c_0 \exp \left\{ C \Phi(v, t) t^{1/2} \right\},
\]

which together with (3.28) yields

\[
\int_0^t \int |x|^\alpha |\nabla \hat{\mathbf{u}}|^2 \, dx \, dt \leq C c_0^2 \exp \left\{ C \Phi(v, t) t^{1/2} \right\}.
\]

Moreover, by virtue of (3.7), we have for \( p \geq 2 \),

\[
(\| \nabla h^2 \|_{L^p})' \leq C\| \nabla v \|_{L^\infty} \| \nabla h^2 \|_{L^p} + C\| h^2 \|_{L^\infty} \| \nabla^2 v \|_{L^p},
\]

which together with (3.9) yields that for \( p \in [2, q] \),

\[
\sup_{0 \leq s \leq t} \| \nabla h \|_{L^p} \leq C \exp \left\{ C \int_0^t \| \nabla v \|_{L^\infty} \, ds \right\} \left( \| \nabla h_0^2 \|_{L^p} + \int_0^t \| h^2 \|_{L^\infty} \| \nabla^2 v \|_{L^p} \, ds \right) \leq C c_0 \exp \left\{ C \Phi(v, t) t^{1/2} \right\}.
\]

Similarly, we deduce from (1.3) and (3.9) that

\[
\sup_{0 \leq s \leq t} \| \nabla h \|_{L^p} \leq C \exp \left\{ C \Phi(v, t) t^{1/2} \right\}.
\]

Noticing that \( \mathbf{u} \) satisfies

\[
\begin{aligned}
\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla (\text{div } \mathbf{u}) &= h \hat{\mathbf{u}} + \nabla h^2 + \mathbf{h} \mathbf{u}^\perp \quad \text{in } \Omega, \\
\mathbf{u} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

we thus derive from the standard \( L^p \) estimate for elliptic equations (see [18]) that for \( p \in (1, \infty) \),

\[
\| \nabla^2 \mathbf{u} \|_{L^p} \leq C (\| h \hat{\mathbf{u}} \|_{L^p} + \| \nabla h^2 \|_{L^p} + \| \mathbf{h} \mathbf{u}^\perp \|_{L^p}),
\]

(3.32)
which together with (3.29)–(3.31), (3.9), and (2.1) yields that

\[
\int_0^t \| \nabla^2 u \|_{L^2}^2 dt \\
\leq C \int_0^t \left( \| h \dot{\nabla} u \|_{L^2}^2 + \| h \nabla^2 u \|_{L^2}^2 + \| h u^\perp \|_{L^2}^2 \right) dt \\
\leq C \int_0^t \left( \| h \|_{L^\infty} \int (1 + |x|)^{\alpha} (|\nabla \dot{\nabla} u|^2 + |\nabla u|^2) dx + \| h \nabla^2 u \|_{L^2}^2 \right) dt \\
\leq C c_0^3 \exp \left\{ C \Phi^2 (v, t) t^{(q-2)/(2(q-1))} \right\}.
\]

Then, it follows from (3.32), (3.5), (3.27), (3.31), and (3.9) that

\[
\sup_{0 \leq s \leq t} \| \nabla^2 u \|_{L^2}^2 \leq C c_0^3 \exp \left\{ C_0 \exp \left\{ C \Phi (v, t) \right\} t^{(q-2)/(2(q-1))} \right\}. \tag{3.33}
\]

**Remark 3.1.** The estimates in Lemmas 3.3 and 3.4 are substantial when investigating the Cauchy problem in two dimensions. It is quite different for 3D problems, in which the $L^6$-norm of $u$ is bounded by the $L^2$-norm of $\nabla u$ directly from the standard Sobolev inequality (see [2]). Also, the estimates hold when the initial height vanishes at far field. In particular, the initial height can even have compact support.

Finally, Lemma 3.5 will close our arguments on the uniform a priori estimates for $\tilde{h} = 0$.

**Lemma 3.5.** Assume that $(h, u)$ is as in Lemma 3.2. Then there exists a time $T^* \in (0, 1]$ depending only on $c_0, \mu, \lambda, \alpha$ such that

\[
\Phi (u, T^*) \leq M,
\]

provided $\Phi (v, T^*) \leq M$ with some given $M = M(\mu, c_0) > 1$.

**Proof.** Lemmas 3.2–3.4 imply that for $q = 4/\alpha$,

\[
\Phi (u, t) \leq C c_0^3 \exp \left\{ C_0 \exp \left\{ C \Phi (v, t) \right\} t^{(q-2)/(2(q-1))} \right\},
\]

which yields that

\[
\Phi (u, T^*) \leq M
\]

by choosing $M = C c_0^3 e^{C c_0}$ and

\[
T^* = \min \left\{ e^{-2C M(q-1)/(q-2)}, 1 \right\}. \tag{3.34}
\]

**3.2. Local existence of strong solutions.** In this subsection, based on the uniform estimates in section 3.1 for solutions to the linearized problem, we use the fixed point theory to show that the shallow water equations (1.3)–(1.5) have unique local strong solutions $(h, u)$.

**Lemma 3.6.** Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Suppose that the initial data $(h_0, u_0)$ satisfy

\[
\begin{align*}
&u_0 \in D^1 \cap D^3, \quad (h_0 - \tilde{h}, h_0^2 - \tilde{h}^2) \in H^3, \\
&h_0 \in H^2, \quad h_0^\perp \in H^1, \quad h_0^\perp + \| h_0 \|_{L^\infty} \leq C \Phi (v, t).
\end{align*}
\]


\( h_0 \geq \delta \) on \( \Omega \) for some \( \delta > 0 \) and the compatibility condition (1.6). In addition, we assume that \( h = 0 \), and

\[
|x|^{\alpha/2} \nabla u_0 \in L^2, \quad |x|^{\alpha/2} h_0^2 \in L^2, \quad |x|^{\alpha/2} h_0^{1/2} g \in L^2
\]

for \( \alpha \) as in (3.3). Then for \( T^* \) as in Lemma 3.5, there exists a unique strong solution \((h, u)\) to the problem (1.3) on \( \Omega \times [0, T^*] \) with

\[
(h, u)|_{t=0} = (h_0, u_0), \quad u|_{\partial \Omega} = 0,
\]

satisfying

\[
\begin{cases}
(h, h^2) \in L^\infty(0, T^*; W^{1,p}), & (h, h^2)_t \in L^\infty(0, T^*; L^p), \\
u \in C([0, T^*]; L^q \cap D^1 \cap D^2) \cap L^2(0, T^*; D^{2,q}), \\
u_t \in L^2(0, T^*; L^q \cap D^1), & \sqrt{h} \nu_t \in L^\infty(0, T^*; L^2)
\end{cases}
\]

for \( q = 4/\alpha \) and any \( p \in [2, q] \). Moreover, the following estimate holds:

\[
\begin{aligned}
\sup_{0 \leq t \leq T^*} \left( \int_{\Omega} h|u|^2 dx + \|
abla^2 u\|_{L^2(\Omega)} + \|(h, h^2)\|_{W^{1,p}(\Omega)} + \|(h, h^2)_t\|_{L^p(\Omega)} \right) \\
+ \sup_{0 \leq t \leq T^*} \int_{\Omega} (1 + |x|^{\alpha}) \left( |\nabla u|^2 + h^4 + h|\dot{u}|^2 + h|u|^2 \right) dx \\
+ \int_{0}^{T^*} \int_{\Omega} (1 + |x|^{\alpha}) |\nabla \dot{u}|^2 dx dt + \int_{0}^{T^*} \|\nabla^2 u\|^2_{L^2(\Omega)} dt \\
\leq C(\mu, \lambda, c_0).
\end{aligned}
\]

**Remark 3.2.** After combing the uniform estimate (3.39) with Caffarelli–Kohn–Nirenberg inequality and some approximations, we bound the \( L^4 \)-norm of \( u \) and \( u_t \) in both bounded and unbounded domain in \( \mathbb{R}^2 \); see Proposition 3.7. Therefore, the higher regularity of \( u_t \) and, consequently, the classical solution \((h, u)\) in \( \mathbb{R}^2 \) can be expected; see (3.48) and Remark 3.3.

**Proof.** For \( T^* \) and \( M \) as in Lemma 3.5, set

\[
\mathcal{B} := L^2(0, T^*; H^1_0)
\]

and

\[
\mathcal{R} := \{ v \in L^\infty(0, T^*; H^1_0 \cap D^2) \cap L^2(0, T^*; D^{2,q}) | \nu_t \in \mathcal{B}, \Phi(v, T^*) \leq M \}.
\]

It is easy to see from Lemma 2.2 that \( \mathcal{R} \) is a convex and compact subset of the Banach space \( \mathcal{B} \). For any \( v \in \mathcal{R} \), by virtue of Lemma 2.5, there exists a unique solution \( h = h(v) \) solving (2.4) on \( \Omega \times [0, T^*] \) and satisfying (2.6). Moreover, according to Lemma 2.6, there exists a unique solution \( u = T(v, h(v)) \) solving the problem (2.9) on \( \Omega \times [0, T^*] \). Here, \((h, u)\) satisfies the a priori estimates stated in Lemmas 3.2–3.5, which yield that \( T \) maps \( \mathcal{R} \) to \( \mathcal{R} \). Next, we will show \( T \) is a continuous operator in \( \mathcal{B} \).
By (3.25) and (2.4), we have
(3.40) \[ \sup_{0 \leq t \leq T^*} \left( \| h(\mathbf{v}) \|_{W^{1,q}} + \| (h(\mathbf{v}))_t \|_{L^q} \right) \leq C(\mu, c_0). \]
Let \( \mathbf{v}_n \in \mathbb{R}, n = 1, 2, \ldots, \) converge to \( \mathbf{v} \) in \( \mathcal{B} \), that is,
(3.41) \[ \mathbf{v}_n \to \mathbf{v} \quad \text{in} \quad L^2(0, T^*; H^1_0) \quad \text{as} \quad n \to +\infty, \]
which together with \( \mathbf{v}_n \in \mathbb{R} \) implies
(3.42) \[ \mathbf{v}_n \to \mathbf{v} \quad \text{in} \quad L^\infty(0, T^*; H^1_0 \cap D^2) \cap L^2(0, T^*; D^2) \quad \text{as} \quad n \to +\infty. \]
It thus follows from Lemma 2.2 and (3.40) that up to a subsequence,
(3.43) \[ h(\mathbf{v}_n) \to h \quad \text{in} \quad C([\Omega \times [0, T^*]) \quad \text{as} \quad n_j \to +\infty. \]
Taking the limits in (2.4), where \( h, \mathbf{v} \) are replaced by \( h(\mathbf{v}_n) \) and \( \mathbf{v}_n \), respectively, we obtain that \( h \) is a weak solution to (2.4), and then, by the uniqueness of the weak solutions due to (3.42) and (3.43), \( h = h(\mathbf{v}) \), which again together with (3.43) and (3.40) implies
(3.44) \[ h(\mathbf{v}_n) \to h \quad \text{in} \quad C([\Omega \times [0, T^*]) \quad \text{as} \quad n \to +\infty \]
and
(3.45) \[ h(\mathbf{v}_n) \to h \quad \text{in} \quad L^\infty(0, T^*; W^{1,q}) \quad \text{as} \quad n \to +\infty. \]
Denoting by \( \mathbf{u}_n = T(v_n, h(\mathbf{v}_n)) \), by virtue of Lemmas 3.2–3.5 and Lemma 2.2, we get\( \mathbf{u}_n \to \mathbf{u} \quad \text{in} \quad L^\infty(0, T^*; H^1_0 \cap D^2) \cap L^2(0, T^*; D^2) \quad \text{as} \quad n \to +\infty \]
and
(3.47) \[ \mathbf{u}_n \to \mathbf{u} \quad \text{in} \quad \mathcal{B} \quad \text{as} \quad n \to +\infty. \]
Letting \( n \to +\infty \) in (2.9), where \( h, \mathbf{u}, \mathbf{v} \) are replaced by \( h(\mathbf{v}_n), \mathbf{u}_n, \mathbf{v}_n \), respectively, we obtain that \( \mathbf{u} \) is a weak solution to (2.9) by (3.46), (3.45), and (3.42), which again yield the uniqueness of the weak solutions to (2.9), and then \( \mathbf{u} = T(\mathbf{v}, h(\mathbf{v})) \), which implies (3.47) holds for \( \mathbf{u}_n \) itself. That means \( T \) is a continuous operator in \( \mathcal{B} \). By the Schauder fixed point theory, Lemma 2.3, there exists some \( \mathbf{u} \in \mathbb{R} \) such that \( T(\mathbf{u}, h(\mathbf{u})) = \mathbf{u} \), which together with Lemmas 3.2–3.4 implies (3.39). Then the uniqueness of \( \mathbf{u} \) and (3.38) follow from (3.39), standard Sobolev embedding theory, and Lemma 2.1.

Now, we can prove the local existence of unique strong solution \( (h, u) \) to the Cauchy problem (1.3)–(1.5).

Proposition 3.7. For \( h \geq 0, \) assume that the initial data \( (h_0 \geq 0, u_0) \) satisfy (3.35) and (1.6). Moreover, if \( h = 0, \) in addition to (3.35) and (1.6), we assume that (3.36) holds. Then there exist a small time \( T^* > 0 \) and a unique strong solution \( (h, u) \) to the Cauchy problem (1.3), (1.4), (1.5) such that

(3.48) \[
\begin{align*}
(h - \bar{h}, h^2 - \bar{h}^2) & \in L^\infty(0, T^*; H^3), \\
(h, h^2)_t & \in L^\infty(0, T^*; H^1), (h, h^2)_{tt} \in L^2(0, T^*; L^2), \\
\mathbf{u} & \in C([0, T^*]; L^2 \cap D^1 \cap D^2) \cap L^2(0, T^*; D^2), \\
\mathbf{u}_t & \in L^\infty(0, T^*; D^1) \cap L^2(0, T^*; L^3 \cap D^2), \\
\sqrt{h} \mathbf{u}_t & \in L^\infty(0, T^*; L^2), \\
\sqrt{h} \mathbf{u} & \in L^\infty(0, T^*; L^2), \\
i \sqrt{h} \mathbf{u}_t & \in L^\infty(0, T^*; L^2), \\
t \mathbf{u}_t & \in L^2(0, T^*; D^1),
\end{align*}
\]
where

\[ q = 2 \quad \text{if} \quad \tilde{h} > 0; \quad q = \frac{4}{\alpha} \quad \text{if} \quad \tilde{h} = 0. \]

**Remark 3.3.** As we will see in the next subsection, the unique strong solution obtained in Proposition 3.7 becomes a classical one, as long as

\[ t^{1/2}u \in L^\infty(0, T^*; D^4), \quad tu_t \in L^\infty(0, T^*; L^q \cap D^2) \]

for some \( q \geq 2 \).

**Proof.** First, we consider the case \( \tilde{h} = 0 \). Define

\[ B_R = \{ x | |x| < R \} \text{ and } \psi(x) = \psi(x/R), g(x) = g(x), h_0 = h_0 + R^{-2} \]

for \((t, x) \in [0, T^*] \times \mathbb{R}^2\), where \( 0 \leq \psi \in C^\infty_0(B_1) \) is a smooth cut-off function such that \( \psi = 1 \) in \( B_{1/2} \).

Let \( u^R_0 \in H^1_0(B_R) \cap H^3(B_R) \) be the unique solution to the elliptic boundary value problem

\[ Lu^R_0 = F^R_0 \quad \text{in} \quad B_R \quad \text{and} \quad u^R_0 |_{\partial B_R} = 0, \]

where

\[ F^R_0 = -\nabla (h_0^R)^2 + \sqrt{h_0^R} g - h_0 u^0. \]

Extending \( u^R_0 \) to \( \mathbb{R}^2 \) by defining zero outside \( B_R \), we can show that

\[ u^R_0 \rightarrow u_0 \quad \text{in} \quad D^1(B_R) \quad \text{as} \quad R \rightarrow \infty. \]

The proof is similar to that in [6], so we omit it here. By Lemma 3.6, there exists some \( T^* > 0 \) such that for each \( R > 1 \), the initial boundary value problem (1.3), (3.37), with \( \Omega = B_R \) and \( (h_0, u_0) = (h^R_0, u^R_0) \) for \( h^R_0, u^R_0 \) defined as above, has a unique strong solution \((h, u)\) satisfying (3.38) and (3.39). We denote such \((h, u)\) by \((h^R, u^R)\). Extending \((h^R, u^R)\) to \( \mathbb{R}^2 \) by zero outside \( B_R \), we have from (3.39) that for \( p \in [2, q] \),

\[
\sup_{0 \leq t \leq T^*} \| \psi R^R_t \|_{W^{1, p}}^2 \\
\leq C \sup_{0 \leq t \leq T^*} \int_{B_R} (h^R)^p dx + C \sup_{0 \leq t \leq T^*} \int_{B_R} |\nabla h^R|^p dx \\
\leq C
\]

and

\[
\sup_{0 \leq t \leq T^*} \int_{\mathbb{R}^2} |(\psi R^R_t)_t|^p dx \leq C \sup_{0 \leq t \leq T^*} \int_{B_R} |h^R_t|^p dx \leq C.
\]
Similarly, (3.39) gives
\[ \sup_{0 \leq t \leq T^*} \left\| \nabla (\psi R \u^R) \right\|_{L^q(B_R)}^2 + \sup_{0 \leq t \leq T^*} \left\| \psi R \u^R \right\|_{L^q(B_R)}^2 \]
\[ \leq C \sup_{0 \leq t \leq T^*} \int_{B_R} |\nabla^2 \psi R|^2 |\u^R|^2 dx + C \sup_{0 \leq t \leq T^*} \int_{B_R} |\nabla \psi R|^2 |\nabla \u^R|^2 dx \]
\[ + C \sup_{0 \leq t \leq T^*} \int_{B_R} |\psi R|^2 |\nabla^2 \u^R|^2 dx + C \sup_{0 \leq t \leq T^*} \int_{B_R} |\psi R|^2 |\nabla \u^R|^2 dx \]
\[ + C \sup_{0 \leq t \leq T^*} \int_{B_R} |\nabla \psi R|^2 |\u^R|^2 dx + C \sup_{0 \leq t \leq T^*} \|\u^R\|_{L^q(B_R)}^2 \]
\[ \leq CR^{-2} \sup_{0 \leq t \leq T^*} \int_{B_R} |\nabla \u^R|^2 dx + C \sup_{0 \leq t \leq T^*} \|\nabla \u^R\|_{L^q(B_R)}^2 \]
\[ \leq CR^{-2+2(q-2)/q} \sup_{0 \leq t \leq T^*} \|\u^R\|_{L^q(B_R)}^2 + C \sup_{0 \leq t \leq T^*} \|\u^R\|_{L^q(B_R)}^2 + C \]
\[ \leq C \sup_{0 \leq t \leq T^*} \int_{B_R} |x|^n |\nabla \u^R|^2 dx + C \]
\[ \leq C \]
and
\[ \int_0^{T^*} \left\| (\psi R \u^R)_t \right\|_{L^q(B_R)}^2 dt \]
\[ \leq C \int_0^{T^*} \|\u_t^R\|_{L^q(B_R)}^2 dt \]
\[ \leq C \int_0^{T^*} \|\u_t^R\|_{L^q(B_R)}^2 dt + C \int_0^{T^*} \|\u^R \cdot \nabla \u^R\|_{L^q(B_R)}^2 dt \]
\[ \leq C \]
which together with (3.39) yields
\[ \int_0^{T^*} \left\| \nabla (\psi R \u^R) \right\|_{L^2(B_R)}^2 dt \]
\[ \leq \int_0^{T^*} \left\| \nabla \psi R \right\|_{L^2(B_R)}^2 dt + \int_0^{T^*} \left\| \psi R \nabla \u^R \right\|_{L^2(B_R)}^2 dt \]
\[ \leq CR^{-2} \int_0^{T^*} \|\u_t^R\|_{L^2(B_R)}^2 dt + \int_0^{T^*} \|\nabla \u^R\|_{L^2(B_R)}^2 dt \]
\[ + \int_0^{T^*} \|\nabla (\u^R \cdot \nabla \u^R)\|_{L^2(B_R)}^2 dt \]
\[ \leq CR^{-4/q} \int_0^{T^*} \|\u_t^R\|_{L^q(B_R)}^2 dt + C \int_0^{T^*} \|\nabla \u^R\|_{L^q(B_R)}^4 dt \]
\[ + C \int_0^{T^*} \|\nabla \u^R\|_{L^q(B_R)}^2 \|\nabla^2 \u^R\|_{L^2(B_R)}^2 dt + C \]
\[ \leq C. \]

Therefore, by Lemma 2.2 and by a diagonalization procedure, we can choose a sub-

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sequence of \( (h^R, u^R) \), still denoted by \( (h^R, u^R) \), such that as \( R \to \infty \)

\[
\psi^R h^R \to h \quad \text{in} \quad L^\infty(0, T^*; W^{1,q}(\mathbb{R}^2)) \cap W^{1,\infty}(0, T^*; L^p(\mathbb{R}^2)),
\]

\[
\psi^R u^R \to u \quad \text{in} \quad C(\bar{B}_n \times [0, T^*]),
\]

\[
\psi^R u^R \to u \quad \text{in} \quad L^\infty(0, T^*; D^1(\mathbb{R}^2) \cap D^2(\mathbb{R}^2)),
\]

\[
\sup_{0 \leq t \leq T^*} \| \nabla u \|_{L^2(\mathbb{R}^2)} \leq C.
\]

and

\[
\psi^R u^R \to u \quad \text{in} \quad C(\mathbb{R}^2 \times [0, T^*]; W^{1,q}(B_n))
\]

for any positive integer \( n \). Now, for any function \( \phi \in C_0^\infty(\mathbb{R}^2 \times [0, T^*]) \), we take \( (\psi^R)^r \phi \) with \( r \geq 4 \) as a test function in (1.3), (3.37). Then letting \( R \to \infty \), it is easy to check that (3.49)–(3.53) yield that \((h, u)\) is a unique strong solution of (1.3), (1.4), (1.5) on \( \mathbb{R}^2 \times [0, T^*] \) satisfying (3.38) with \( \Omega = \mathbb{R}^2 \). In the case \( \tilde{h} > 0 \), we can bound \( \|u\|_{L^\infty(0, T^*;L^2(\mathbb{R}^2))} \) and \( \|\nabla u\|_{L^2(0, T^*;L^2(\mathbb{R}^2))} \) according to Lemma 3.1. Then, it is easy to get the unique strong solution as in [2]. In order to show (3.48), we can follow the steps in section 3 of [20] and obtain the higher order estimates of the solutions since \( \|u\|_{L^\infty(0, T^*;L^2(\mathbb{R}^2))} \) and \( \|\nabla u\|_{L^2(0, T^*;L^2(\mathbb{R}^2))} \) are bounded, where \( q = 2 \) if \( \tilde{h} > 0 \) and \( q = 4/\alpha \) if \( \tilde{h} = 0 \). The proof of Proposition 3.7 is completed.

### 3.3. Regularity analysis

In this subsection, we will show the necessary higher regularity for the classical solution and then prove our main result. We remark here that for arbitrary data, which may include vacuum states, the following lemma is substantial for the regularity of solution. Here, we estimate the spatial weighted norm of suitable quantities to overcome the difficulties coming from the space dimension and the appearance of vacuum. Moreover, an elaborate spatial weight is adopted in order to close the estimates when using the energy estimate method.

**Lemma 3.8.** If \( \tilde{h} = 0 \), the following estimate holds:

\[
\sup_{0 \leq t \leq T^*} \| \nabla u \|_{L^2(\mathbb{R}^2)} \leq C.
\]

**Proof.** Setting \( v = u \) in (3.26) and multiplying the resulting equation by \(|x|^{\alpha/2}(\tilde{u})_t\), we obtain after integrating by parts that

\[
\int |x|^{\alpha/2} h(\tilde{u})_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \left( \mu \int |x|^{\alpha/2} |\nabla \tilde{u}|^2 \, dx + (\mu + \lambda) \int |x|^{\alpha/2} |\text{div} \tilde{u}|^2 \, dx \right)
\]

\[
= - \int h(u \cdot \nabla \tilde{u}) |x|^{\alpha/2}(\tilde{u})_t \, dx - \int h(x|^{\alpha/2} (\tilde{u})_t \cdot (\tilde{u})_t \, dx
\]

\[
- \frac{\mu \alpha}{2} \int |\partial_k \tilde{u}^k \cdot (\tilde{u})_t |x|^{\alpha/2-2} x^i \, dx - \frac{\alpha(\mu + \lambda)}{2} \int |\text{div} \tilde{u}((\tilde{u})_t \cdot x)|^{\alpha/2-2} \, dx
\]

\[
+ \mu \int (\partial_k \tilde{u}^k \partial_t \partial_i |x|^{\alpha/2}(\tilde{u})_t \, dx - \mu \int (\partial_k \tilde{u}^k \partial_t \partial_i |x|^{\alpha/2}(\tilde{u})_t \, dx
\]

\[
+ \mu \int (\partial_k \tilde{u}^k \partial_t |x|^{\alpha/2}(\tilde{u})_t \, dx
\]

\[
- (\mu + \lambda) \left[ \int |\text{div} u|^2 \, dx - \int |(\partial_k \tilde{u}^k \partial_t u)|^2 \, dx - \int |\nabla u^k \cdot \partial_k (|x|^{\alpha/2}(\tilde{u})_t |x|^{\alpha/2}(\tilde{u})_t \, dx\right]
\]

\[
- \int h^2 \nabla u^k \cdot \partial_k (|x|^{\alpha/2}(\tilde{u})_t \, dx - \int h^2 \text{div} u |x|^{\alpha/2} (\tilde{u})_t \, dx.
\]
We use (2.1), (3.39), and (3.48) to estimate each terms on the right-hand side of (3.55) as follows:

\[
\begin{align*}
\int h(u \cdot \nabla \hat{u}) |x|^\alpha/2 (\hat{u})_t dx + \int h|x|^\alpha/2 (\hat{u})_t \cdot (\hat{u})_t dx \\
\leq C \|h^{1/2}(\hat{u})_t| |x|^\alpha/4 \|_{L^2} \left( \|u\|_{L^\infty} \|\nabla \hat{u}\|_{L^2} + \|h^{1/2} \hat{u}x^{\alpha/4} \|_{L^2} \right) \\
\leq \frac{1}{2} \|h^{1/2}(\hat{u})_t| |x|^\alpha/4 \|_{L^2}^2 + C \|\nabla \hat{u}\|_{L^2} \|\nabla \hat{u}\|_{L^2} + C \|h^{1/2} \hat{u}x^{\alpha/2} \|_{L^2}^2 \\
+ C \|h^{1/2} \hat{u}\|_{L^2}^2 \\
\leq \frac{1}{2} \|h^{1/2}(\hat{u})_t| |x|^\alpha/4 \|_{L^2}^2 + C \|\nabla \hat{u}\|_{L^2} \|\nabla \hat{u}\|_{L^2} + C(\|\nabla \hat{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|h^{1/2}u_t\|_{L^2}^2);
\end{align*}
\]

next,

\[
- \frac{\mu \alpha}{2} \int \partial_t \hat{u}^k (\hat{u}^k)_t |x|^\alpha/2 - 2 x^i dx \\
\leq - \frac{\mu \alpha}{2} \frac{d}{dt} \int \partial_t \hat{u}^k \hat{u}^k |x|^\alpha/2 - 2 x^i dx + C \int |\nabla (\hat{u})_t| \|\hat{u}\|_{L^2} |x|^{\alpha/2 - 2} dx \\
\leq - \frac{\mu \alpha}{2} \frac{d}{dt} \int \partial_t \hat{u}^k \hat{u}^k |x|^\alpha/2 - 2 x^i dx \\
+ C \|\nabla (u_t + u \cdot \nabla u + u \cdot \nabla u)_t\|_{L^2} \|\hat{u}\|_{L^2} |x|^{\alpha/2 - 2} dx \\
\leq - \frac{\mu \alpha}{2} \frac{d}{dt} \int \partial_t \hat{u}^k \hat{u}^k |x|^\alpha/2 - 2 x^i dx \\
+ C(\|\nabla \hat{u}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + 1).
\]

Also, we can estimate the terms

\[
\begin{align*}
\mu \int (\partial_t u^k \partial_x u) \partial_t (|x|^\alpha/2 (\hat{u})_t) dx \\
\leq \mu \frac{d}{dt} \int (\partial_t u^k \partial_x u) \partial_t (|x|^\alpha/2 \hat{u}) dx \\
+ C \int |\nabla u_t| |\nabla u| (|\nabla \hat{u}| |x|^\alpha/2 + |\hat{u}| |x|^{\alpha/2}) dx \\
\leq \mu \frac{d}{dt} \int (\partial_t u^k \partial_x u) \partial_t (|x|^\alpha/2 \hat{u}) dx + C \|\nabla \hat{u}\|_{L^2}^2 + C
\end{align*}
\]

and

\[
\begin{align*}
- \int (h^2 \nabla u^k) \cdot \partial_t (|x|^\alpha/2 (\hat{u})_t) dx \\
\leq - \frac{d}{dt} \int (h^2 \nabla u^k) \cdot \partial_t (|x|^\alpha/2 \hat{u}) dx \\
+ C \int (|h^2| |\nabla u| + h^2 |\nabla u_t|) (|\nabla \hat{u}| |x|^\alpha/2 + |\hat{u}| |x|^{\alpha/2}) dx \\
\leq - \frac{d}{dt} \int (h^2 \nabla u^k) \cdot \partial_t (|x|^\alpha/2 \hat{u}) dx + C \|\nabla \hat{u}\|_{L^2}^2 + C
\end{align*}
\]
It follows that in the same way as (3.57)–(3.59), we can estimate the other terms on the right-hand side of (3.55). Therefore, we arrive at\[\begin{align*}
\frac{1}{2} \int |x|^{\alpha/2} h \dot{u}_i^2dx + \frac{1}{2} \frac{d}{dt} \left( \mu \int |x|^{\alpha/2} |\nabla \dot{u}|^2dx + \right)
\left( \mu + \lambda \right) \int |x|^{\alpha/2} |\text{div} \dot{u}|^2dx \n\leq \frac{d}{dt} \Psi(t) + C(|\nabla \dot{u}|^2_{L^2} + ||\nabla \dot{u}|x|^{\alpha/2}|^2_{L^2} + ||\nabla u_t||^2_{L^2} + ||u_t||^2_{L^{4/\alpha}}) \n+ C||\nabla^2 u||^2_{L^2} + C,
\end{align*}\]
where\[\Psi(t) := -\frac{\mu \alpha}{2} \int \partial_i \dot{u}^k \dot{u}^k |x|^{\alpha/2 - 2} x^i dx - \frac{\alpha (\mu + \lambda)}{2} \int \text{div} \dot{u} (\dot{u} \cdot x) |x|^{\alpha/2 - 2} dx \n+ \mu \int (\partial_i u^k \partial_i u^k) \partial_i (|x|^{\alpha/2} \dot{u}) dx - \mu \int (\partial_i u^k \partial_i u^k) \partial_i (|x|^{\alpha/2} \dot{u}) dx \n+ \mu \int (\partial_i u^k \partial_i u^k) \partial_i (|x|^{\alpha/2} \dot{u}) dx \n- (\mu + \lambda) \left[ \int (\text{div} u)^2 \text{div} (|x|^{\alpha/2} \dot{u}) dx - \int (\partial_i u^k \partial_i u^k) \text{div} (|x|^{\alpha/2} \dot{u}) dx \n- \int (\text{div} u^k) \partial_i (|x|^{\alpha/2} \dot{u}) dx \right] \n- \int (h^2 \nabla^k u^k) \cdot \partial_i (|x|^{\alpha/2} \dot{u}) dx - \int h^2 \text{div} (u^k) \text{div} (|x|^{\alpha/2} \dot{u}) dx.\]

It is easy to show that\[|\Psi(t)| \leq \mu \alpha^2 ||x|^{\alpha/4} \nabla \dot{u}|^2_{L^2} + (\mu + \lambda) \alpha^2 ||x|^{\alpha/4} \nabla \dot{u}|^2_{L^2} \n+ C \int (|\nabla u|^2 + |\nabla u|) \left( |x|^{\alpha/2} |\nabla \dot{u}| + |\dot{u}| |x|^{(\alpha - 2)/2} \right) dx \n\leq (2\mu + \lambda) \alpha^2 ||x|^{\alpha/4} \nabla \dot{u}|^2_{L^2} + C ||\nabla u||^{\alpha/4} ||x||^{\alpha/\alpha} ||\nabla \dot{u}|^2_{L^2} \n\leq 2(2\mu + \lambda) \alpha^2 ||x|^{\alpha/4} \nabla \dot{u}|^2_{L^2} + C ||\nabla u||^{\alpha/2} ||x||^{\alpha/2} + C ||\nabla u||^2_{L^2} \n\leq \frac{\mu}{8} ||x|^{\alpha/4} \nabla \dot{u}|^2_{L^2} + C.\]

Multiplying (3.60) by t and integrating the resulting equation over (0, T*), we deduce from (3.61), (3.48), and (3.39) that\[\begin{align*}
\sup_{0 \leq t \leq T^*} \int t |x|^{\alpha/2} |\nabla \dot{u}|^2 dx + \int_0^{T^*} t \int |x|^{\alpha/2} h |\dot{u}_t|^2 dx dt \n\leq C \int_0^{T^*} \left( ||\nabla \dot{u}|^2_{L^2} + ||\nabla \dot{u}|x|^{\alpha/2}|^2_{L^2} + t ||\nabla u_t||^2_{L^2} + ||u_t||^2_{L^{4/\alpha}} \right) dt \n+ C \int_0^{T^*} (||\nabla^2 u||^2_{L^2} + 1) dt \n\leq C.
\end{align*}\]

It follows from (3.62) and (2.1) that\[\sup_{0 \leq t \leq T^*} t ||\dot{u}_t||^{2}_{L^{8/\alpha}} \leq C \sup_{0 \leq t \leq T^*} t ||x|^{\alpha/4} \nabla \dot{u}|^2_{L^2} \leq C,
\]
which yields that
\begin{equation}
\sup_{0 \leq t \leq T} \|u_t\|_{L^8}^2 \leq C \sup_{0 \leq t \leq T} t \left( \|\dot{u}\|_{L^8}^2 + \|u \cdot \nabla u\|_{L^8}^2 \right) \leq C.
\end{equation}

Next, the standard $L^2$-estimate for the elliptic system, (3.39) and (3.48) yield that for $p \in [q, \infty)$
\[
\|\nabla^2 u\|_{L^2} \leq C\|\mu \Delta u + (\mu + \lambda)\nabla \text{div} u\|_{L^2} = \|h u_t + h_t u + h_\Omega \cdot \nabla u + h u \cdot \nabla u + \nabla (h^2)_t + (h u^\perp)_t\|_{L^2}
\]
\begin{equation}
\leq C \left( \|h u_t\|_{L^2} + \|h_t\|_{L^{2p/(p-2)}} + \|h\|_{L^{2p/(p-2)}} \left( \|u_t\|_{L^p} + \|u\|_{L^p} \right) + \|h_t\|_{L^1} \|u\|_{L^\infty} \|\nabla u\|_{L^1} + \|u_t\|_{L^p} \|\nabla u\|_{L^{2p/(p-2)}} + \|u\|_{L^\infty} \|\nabla u\|_{L^2} + \|\nabla (h^2)_t\|_{L^2} \right)
\end{equation}
\[
\leq C h^{1/2} u_t \|u\|_{L^2} + C \|u_t\|_{L^p} + C.
\]
The standard $L^2$-estimate for the elliptic system again yields that for $p \in [q, \infty)$
\[
\|\nabla^4 u\|_{L^2} \leq C \|\nabla^2 (\mu \Delta u + (\mu + \lambda)\nabla \text{div} u)\|_{L^2} \leq C \|\nabla^2 (h \dot{u})\|_{L^2} + C \|\nabla^3 h^2\|_{L^2} + C \|h u^\perp\|_{L^2}
\]
\begin{equation}
\leq C + C \|\nabla u\|_{H^2} + C \|u_t\|_{L^p} + C \|\nabla u\|_{H^1} + C \|\nabla^3 h^2\|_{L^2},
\end{equation}
where one has used the following simple facts:
\begin{equation}
\|\nabla^2 (h u_t)\|_{L^2} \leq C \left( \|\nabla^2 h\|_{L^2} \|u_t\|_{L^p} + \|\nabla u_t\|_{H^1} \right) + \|\nabla h\|_{L^4} \|\nabla u_t\|_{L^4} + \|\nabla^2 u_t\|_{L^2}
\end{equation}
\[
\leq C \|u_t\|_{L^p} + C \|\nabla u_t\|_{H^1}
\]
and
\begin{equation}
\|\nabla^2 (h u \cdot \nabla u)\|_{L^2} \leq C \left( \|\nabla^2 (h u)\|_{L^2} \|\nabla u\|_{H^2} + \|\nabla (h u)\|_{L^4} \|\nabla^2 u\|_{L^4} + \|\nabla^3 u\|_{L^2} \right)
\end{equation}
\[
\leq C \left( 1 + \|\nabla^2 h\|_{L^2} \|u\|_{L^\infty} + \|\nabla h\|_{L^4} \|\nabla u\|_{L^4} + \|\nabla^2 u\|_{L^2} \right) \|\nabla u\|_{H^2}
\]
due to (3.39), (3.48), and (2.3). Choosing $p = 8/\alpha$ in both (3.64) and (3.65), together with (3.48), yields that
\begin{equation}
\sup_{0 \leq t \leq T} \|\nabla^4 u\|_{L^2} \leq C.
\end{equation}
Thus, (3.54) follows from (3.63) and (3.68) directly. We finish the proof of Lemma 3.8. \(\square\)

We are now in a position to prove our main result, Theorem 3.9.

**Theorem 3.9.** For $h \geq 0$ and $\Omega = \mathbb{R}^2$, assume that the initial data $(h_0 \geq 0, u_0)$ satisfy the condition in Proposition 3.7. Then there exist a small time $T^* > 0$ and a unique solution $(h, u)$ to the Cauchy problem (1.3), (1.4), (1.5) on $\mathbb{R}^2 \times [0, T^*]$ such
that for any $0 < \tau < T^*$,
\[
\begin{align*}
(h - \bar{h}, h^2 - (\bar{h})^2) & \in C([0, T^*]; H^3), \\
u & \in C([0, T^*]; L^2 \cap D^1 \cap D^2) \cap L^2(0, T^*; D^4) \cap L^\infty(\tau, T^*; D^4), \\
u_t & \in L^\infty(0, T^*; D^1) \cap L^2(0, T^*; L^3 \cap D^2) \\
u_{tt} & \in L^\infty(\tau, T^*; L^1 \cap D^2) \cap H^1(\tau, T^*; D^1), \\
\sqrt{\nu}u_t & \in L^\infty(0, T^*; L^2), \quad \sqrt{\nu}u_{tt} \in L^2(0, T^*; L^2), \\
t^{1/2}\sqrt{\nu}u_{tt} & \in L^\infty(0, T^*; L^2),
\end{align*}
\]

where
\[
\begin{cases}
q = q_1 = 2 & \text{if } \bar{h} > 0, \\
q = 4/\alpha, \ q_1 = 8/\alpha & \text{if } \bar{h} = 0.
\end{cases}
\]

Remark 3.4. The unique solution obtained in Theorem 3.9 becomes a classical one for positive time.

Remark 3.5. The local classical solution obtained in Theorem 3.9 exists for arbitrary large initial data and the initial height need not be positive and may vanish in open sets. In particular, the initial height can even have compact support.

Proof. If $\bar{h} = 0$, Theorem 3.9 is an easy consequence of Proposition 3.7 and Lemma 3.8. It remains to prove that $(h, u)$ becomes a classical solution for positive time, that is, for any $0 < \tau \leq T^*$,

\begin{equation}
(3.69) \quad u_t, \nabla^2 u, \nabla h, h_t \in C([\tau, T^*] \times \mathbb{R}^2).
\end{equation}

It follows from (3.48) and (3.54) that for any $0 < \tau \leq T^*$,
\[
\sup_{\tau \leq t \leq T^*} \|\nabla^2 u\|_{H^2} \leq C(\tau),
\]
which, together with (3.48), $(\nabla^2 u)_t \in L^2(\mathbb{R}^2 \times (0, T^*))$, implies that for $p > 2$,

\begin{equation}
(3.70) \quad \nabla^2 u \in C([\tau, T^*]; H^1 \cap W^{1,p}) \hookrightarrow C([\tau, T^*] \times \mathbb{R}^2).
\end{equation}

By virtue of (3.54) and (3.48),

\begin{equation}
(3.71) \quad \sup_{\tau \leq t \leq T^*} \left( \|u_t\|_{L^{8/\alpha}} + \|\nabla u_t\|_{H^1} \right) + \int_\tau^{T^*} \|\nabla u_t\|_{L^2}^2 dt \leq C(\tau),
\end{equation}

which implies for $p > 2$,

\begin{equation}
(3.72) \quad \nabla u_t \in C([\tau, T^*]; L^2 \cap L^p).
\end{equation}

It follows from (3.71) and (3.72) directly that

\begin{equation}
(3.73) \quad u_t \in C([\tau, T^*] \times \mathbb{R}^2).
\end{equation}

We deduce from (3.48) that
\[
\nabla h \in L^\infty(0, T^*; H^2), \quad (\nabla h)_t \in L^\infty(0, T^*; L^2),
\]
which yields that for $p > 2$,

\begin{equation}
(3.74) \quad \nabla h \in C([0, T^*]; H^1 \cap W^{1,p}) \hookrightarrow C([0, T^*] \times \mathbb{R}^2).
\end{equation}
According to (1.3), (3.70), and (3.74), we have
\[ h_t \in C \left([0, T^*] \times \mathbb{R}^2\right), \]
which together with (3.70), (3.73), and (3.74) gives (3.69). If \( \bar{h} > 0 \), after bounding \( \|u\|_{L^2(\mathbb{R}^2)} \) and \( \|\dot{u}\|_{L^2(\mathbb{R}^2)} \) by Lemma 3.1, we can obtain Theorem 3.9 as above. The proof of Theorem 3.9 is completed.

4. Blow-up example. In this section, we investigate the blow-up behavior of 2D smooth radially symmetric solutions to shallow water equations (1.3) when the initial height has compact support. If the initial data \((h_0, u_0)\) are radially symmetric, i.e.,
\[ h_0(x) = h_0(r), \quad u_0(x) = v_0(r) \frac{x}{r}, \quad r = |x|, \]
satisfying conditions (3.35), (1.6), and (3.36), the shallow water equations has unique local radially symmetric solutions \((h, u)\),
\[ h(x, t) = h(r, t), \quad u(x, t) = v(r, t) \frac{x}{r}. \]

Now, following the idea in [39], we can state the blow-up result on smooth solutions when the initial data \((h_0, u_0)\) are radially symmetric and the initial height is compactly supported.

**Theorem 4.1.** For \( \bar{h} = 0 \), suppose that \((h_0 \neq 0, u_0)\) satisfy (4.1) and that the nonnegative initial height \(h_0\) has compact support. Then any radially symmetric solution \((h, u)\) is \( C^1([0, T]; H^s) \) for any \( s \geq 2 \) and initial data \((h_0, u_0)\) has to blow up in finite time, that is, \( T \) must be finite.

**Proof.** Since the support of the initial radially symmetric height \(h_0\) is compact, we can assume that
\[ \text{supp} h_0 = B_{R_0}(0) = \{ x \in \mathbb{R}^2 | x | \leq R_0 \} \]
for some \( R_0 > 0 \). We denote by \( X(t; x_0) \) the particle path starting at \( x_0 \) when \( t = 0 \), i.e.,
\[ \frac{d}{dt} X(t, x_0) = u(X(t, x_0), t), \quad X(t = 0; x_0) = x_0. \]

Set
\[ S_p(t) := \{ x \in X(t; x_0) \cap B_{R_0}(0) \} \]
It follows from the continuity equation that the height is transported along particle paths, so that
\[ \text{supp}_{x} h(x, t) \subseteq S_p(t). \]
Consequently, \( h = 0 \) on \( \{t\} \times (\mathbb{R}^2 \setminus S_p(t)) \) and one has from the momentum equation and (4.2) that
\[ (2\mu + \lambda) \partial_r \left( v_r + \frac{\dot{u}}{r} \right) = 0 \quad \text{on} \quad \{t\} \times (\mathbb{R}^2 \setminus S_p(t)), \]
since $\Delta u(x, t) = \nabla \text{div } u(x, t) = \partial_t (v_r + \frac{\gamma}{2} x)$. Therefore,

$$v(r, t) = \theta(t)r + \frac{\beta(t)}{r} \quad \text{on } \{t\} \times (\mathbb{R}^2 \setminus S_p(t)).$$

Now, because $u \in C^1([0, T]; H^s)(s > 2)$, we have $u(x, t) \equiv 0$ in $x \in \mathbb{R}^2 \setminus S_p(t)$, and

$$S_p(t) = B_{R_0}(0), \quad \forall 0 < t < T.$$

Following [39], we denote

$$I_2 = \int x^2 h dx - 2(1 + t) \int h u \cdot x dx$$

Then

$$\frac{d}{dt} I_2(t) = \int (x^2 h_t - 2 h u \cdot x) dx - 2(1 + t) \int \left[ (h u)_t \cdot x - h u^2 - \frac{2}{\gamma - 1} h^\gamma \right] dx$$

Direct calculation by using (4.2), (4.4), and $u \in C^1([0, T]; H^s)(s > 2)$ yields that $I_1 = 0$,

$$I_2 = -2(1 + t) \int \left[ -\text{div}(h u \otimes u) \cdot x - \nabla h^\gamma \cdot x - h u^\perp \cdot x + \mu \Delta u \cdot x + (\mu + \lambda)(\nabla \text{div } u) \cdot x \right] dx + 2(1 + t) \int h u^2 + \frac{2}{\gamma - 1} h^\gamma dx$$

$$= -2(1 + t) \left[ \int \left( 2 - \frac{2}{\gamma - 1} \right) h^\gamma dx + (2\mu + \lambda) \int_0^{+\infty} \partial_r \left( \frac{v}{r} + \nu_r \right) \left( \frac{|x|^2}{r} \right) r dr \right]$$

and

$$I_3 = (1 + t)^2 \int 2 \left[ -\nabla h^\gamma \cdot u + \mu \Delta u \cdot u + (\mu + \lambda)(\nabla \text{div } u) \cdot u \right] dx$$

$$+ (1 + t)^2 \frac{2}{\gamma - 1} \int [-\text{div}(h^\gamma u) - (h^\gamma)' h - h^\gamma \text{div } u] dx$$

$$= 2(1 + t)^2 \int_0^{+\infty} (2\mu + \lambda) \partial_r \left( \frac{v}{r} + \nu_r \right) \frac{x}{r} \cdot \left( \frac{x}{r} \right) r dr$$

$$= -2(1 + t)^2(2\mu + \lambda) \int \left( \frac{v}{r} + \nu_r \right)^2 dx.$$
Therefore,

\[
I'_\gamma(t) = 2(2\mu + \lambda) \left[ 2(1+t) \int \left( \frac{v}{r} + v_r \right) dx - (1+t)^2 \int \left( \frac{v}{r} + v_r \right)^2 dx \right] \\
+ \frac{4(1+t)(2-\gamma)}{\gamma - 1} \int \tilde{h}_\gamma dx \\
\leq \frac{4(1+t)(2-\gamma)}{\gamma - 1} \int \tilde{h}_\gamma dx + 2(2\mu + \lambda) \int_{S_p(t)} dx \\
= \frac{4(1+t)(2-\gamma)}{\gamma - 1} \int \tilde{h}_\gamma dx + 2(2\mu + \lambda)|B_{R_0}(0)|.
\]

(4.8)

If \( \gamma \geq 2 \), (4.8) gives

\[
I'_\gamma(t) \leq 2(2\mu + \lambda)|B_{R_0}(0)|,
\]

which yields that

\[
I_{\gamma}(t) \leq I_{\gamma}(0) + 2(2\mu + \lambda)|B_{R_0}(0)|t.
\]

Thus, we have

\[
\int h^\gamma dx \leq \frac{(\gamma - 1)I_{\gamma}(0)}{2} (1+t)^{-2} + 2(2\mu + \lambda)|B_{R_0}(0)|(\gamma - 1)(1+t)^{-1}
\]

(4.9)

due to (4.6). On the other hand, if \( 1 < \gamma < 2 \), let \( \alpha = 2 - 2(\gamma - 1) \). It follows from (4.8) that

\[
(1+t)^{-\alpha}I_{\gamma}(t)' \leq 2(2\mu + \lambda)|B_{R_0}(0)|(1+t)^{2\gamma-4},
\]

(4.10)

which gives

\[
(1+t)^{-\alpha}I_{\gamma}(t) \leq I_{\gamma}(0) + 2(2\mu + \lambda)|B_{R_0}(0)|F(t),
\]

(4.11)

where

\[
F(t) = \begin{cases} 
\frac{(1+t)^{2\gamma-3}}{2\gamma-3} & \text{if } 2\gamma - 3 \neq 0; \\
\ln(1+t) & \text{if } 2\gamma - 3 = 0.
\end{cases}
\]

This yields that

\[
\int h^\gamma dx \leq \frac{(\gamma - 1)I_{\gamma}(0)}{2} (1+t)^{-2(\gamma-1)} + (\gamma - 1)(2\mu + \lambda)|B_{R_0}(0)|F(t)(1+t)^{-2(\gamma-1)}.
\]

(4.12)

Finally, from the continuity equation, (4.5), (4.4), and Hölder’s inequality, we have

\[
0 < \int h_0 dx = \int h dx \leq \left( \int h^\gamma dx \right)^{\frac{1}{\gamma}} |B_{R_0}(0)|^{-\frac{1}{\gamma-1}}, \quad t \in [0,T] .
\]

Since \( \lim_{t \to +\infty} F(t)(1+t)^{-2(\gamma-1)} = 0 \), the above inequality together with (4.9) and (4.12) implies that \( T \) must be finite. We finish the proof of Theorem 4.1. \( \blacksquare \)
Remark 4.1. Following the idea in [39], Theorem 4.1 presents a sufficient condition on the blow-up of smooth solution to the rotating shallow water equations with initial height of compact support. There is a distinctive feature that the development of singularity in the smooth solution in the case away from a vacuum is due to the formation of shocks. However, in the case with compact height, the development of singularity is due to the loss of smoothness near the flow boundary, by which the fluids and the vacuum are separated.

Remark 4.2. According to the blow-up phenomena in Theorem 4.1, when the initial height has compact support, a $C^1([0,T];H^s)(s > 2)$ solution cannot exist globally. Therefore, for rotation shallow water equations, the solutions in our main result, Theorem 3.9, give a possible class, in which the globally in time solutions are expected in the case $\tilde{h} = 0$.

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