GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS TO SHALLOW WATER EQUATIONS WITH CAUCHY DATA CONTAINING VACUUM

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Abstract. In this paper, we investigate the singularity formulation of Cauchy problem to the viscous compressible rotating shallow water equations. We obtain the global existence of classical solutions with initial vacuum. In particular, the spatial measure of the set of vacuum can be arbitrarily large.

2000 Mathematics Subject Classification: 35Q35; 35M10; 76N10; 76U05.

Key words: Viscous compressible rotating shallow water system; Cauchy problem; Global solution; Classical solution

1. Introduction

In this paper, we investigate the nonlinear shallow water equations with following form,

\[
\begin{aligned}
& h_t + \text{div}(hu) = 0, \\
& (hu)_t + \text{div}(hu \otimes u) + gh\nabla h + f(hu)^\perp = \mu \Delta u + (\mu + \lambda)\nabla (\text{div} u),
\end{aligned}
\]

where \( h(x, t) \) is the height of the fluid surface, \( u(x, t) \) is the horizontal velocity field, \( g > 0 \) is the gravity constant, \( f > 0 \) is the Coriolis frequency, \( \mu \) and \( \lambda \) are the dynamical viscosities satisfying

\[
\mu > 0, \quad \mu + \lambda \geq 0.
\]

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Without losing the essential ingredients of (1.1), we focus on the following system for simplicity. 

\[
\begin{aligned}
    h_t + \text{div}(hu) &= 0, \\
    (hu)_t + \text{div}(hu \otimes u) + 2h\nabla h + hu^\perp &= \mu \Delta u + (\mu + \lambda)\nabla(\text{div}u).
\end{aligned}
\]  

(1.2)  

Let \( \Omega = \mathbb{R}^2 \) and \( \tilde{h} \) be a fixed nonnegative constant. We look for the solutions, \((h(x,t), u(x,t))\), to the Cauchy problem for (1.1) with the far field behavior:  

\[
u(x,t) \to 0, \quad h(x,t) \to \tilde{h} \geq 0, \quad \text{as} \ |x| \to \infty,
\]  

(1.3)  

and initial data,  

\[
(h, u)|_{t=0} = (h_0 \geq 0, u_0), \quad x \in \mathbb{R}^2.
\]  

(1.4)  

For the shallow water system (1.1), there are amounts of literature. Assuming the initial data is a small perturbation of a positive constant, Sundbye \[18, 19\] proved global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem, also the Cauchy problem, using Sobolev space estimates and energy method of Matsumura and Nishida \[14, 15, 16\]. Wang and Xu \[21\], obtained local solutions for any initial data and global solutions for small initial data, \( h_0 - \bar{h}_0, u_0 \in H^{2+s}(\mathbb{R}^2) \) with \( \bar{h}_0 \) and \( s > 0 \). The result was improved by Chen, Miao and Zhang in \[14\], they prove the local existence in time for general initial data and global existence in time for small initial data where \( h_0 - \bar{h}_0 \in B^0_{2,1} \cap B^1_{2,1} \) and \( u_0 \in B^0_{2,1} \) with additional condition that \( h \geq \bar{h}_0 > 0 \). A global existence of strong solution in the space of Besov type was obtained by Hao-Hisao-Li \[8\], for the shallow water equations involving the capillary term \( h\nabla \Delta h \), with the initial data close to a positive constant equilibrium state. For arbitrary large initial data, Bresch-Desjardins \[1, 2\] and Bresch-Desjardings-Lin \[3\] proved the global existence of weak solutions for 2D shallow water equations in bounded domain with periodic boundary conditions, where the friction term and the capillary term are involved. Later, Li-Li-Xin showed the global existence of weak entropy solution for the initial boundary value problem to 1D compressible flows. The same result was obtain by Guo-Jiu-Xin \[7\] for the case of multi-dimensional spherically symmetric weak solution.

For global classical solutions, it should be noted that previous work only consider the case of small data. In general, the problem of existence for large initial data is difficult because of its strong non-linear nature. In addition, when investigating solutions with high regularity, no initial vacuum is very crucial. In this paper, by taking the advantages of high regularity estimates of \( \rho \) and \( u \), we improve all previous results for rotating shallow water equations by removing the restriction on small perturbation of a positive constant states, and allowing initial vacuum states.

Before starting the main results, we explain the notations and conventions used throughout this paper. We denote 

\[
\int f \, dx = \int_{\mathbb{R}^2} f \, dx.
\]
For $1 \leq r \leq \infty$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$
\begin{align*}
L^r &= L^r(\mathbb{R}^2), \\
D^{k,r} &= \left\{ u \in L^1_{loc}(\mathbb{R}^2) \mid \| \nabla^k u \|_{L^r} < \infty \right\}, \\
W^{k,r} &= L^r \cap D^{k,r}, \\
H^k &= W^{k,2}, \\
D^k &= D^{k,2}, \\
D^0_0 &= \left\{ u \in D^k \mid u|_{\partial \Omega} = 0 \right\}.
\end{align*}
$$

The initial energy is defined as:

$$
C_0 = \int \left( \frac{1}{2} h_0 |u_0|^2 + G(h_0) \right) \, dx,
$$

where $G(h)$ is given by

$$
G(h) \equiv h \int_0^h \frac{s^2 - \tilde{h}^2}{s^2} \, ds.
$$

It is easy to see that

$$
c_1(\tilde{h}, \bar{h})(h - \tilde{h})^2 \leq G(h) \leq c_2(\tilde{h}, \bar{h})(h - \tilde{h})^2,
$$

for positive constants $c_1(\tilde{h}, \bar{h})$ and $c_2(\tilde{h}, \bar{h})$. Moreover, the material derivative is defined as

$$
\dot{f} \equiv f_t + u \cdot \nabla f.
$$

We begin with local existence of classical solutions, in the absence of vacuum, the local existence and uniqueness are known in $H^3$ using Lagrangian coordinates and Hölder space estimates with initial data in $C^{2+\alpha}$. In the case where the initial height need not be positive and may vanish in an open set, the existence and uniqueness of local classical solutions are provided recently in $H^2$; they also give a blowup example when the initial height have a compact support. In particular, they obtained the following result.

**Theorem 1.1.** For $\bar{h} \geq 0$ and $\Omega = \mathbb{R}^2$, suppose that the initial data $(h_0 \geq 0, u_0)$ satisfy

$$
u_0 \in D^1 \cap D^3, \quad (h_0 - \bar{h}, \bar{h}^2 - \bar{h}) \in H^3,
$$

and the compatibility condition

$$
- \mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla h_0^2 + h_0 u_0^\perp = h_0 g,
$$

for some $g \in D^1$ with $h_0^{1/2} g \in L^2$. In addition, we assume that $\bar{h} = 0$, and

$$
|x|^{\alpha/2} \nabla u_0 \in L^2, \quad |x|^{\alpha/2} h_0^2 \in L^2, \quad |x|^{\alpha/2} h_0^{1/2} g \in L^2,
$$

for $\alpha$ where

$$
\alpha \equiv \mu/(4(2\mu + \lambda)) \in (0,1/8].
$$

Then there exist a small time $T_0 > 0$ and a unique solution $(h,u)$ to the Cauchy problem on $\mathbb{R}^2 \times [0,T_0]$ such that for any $0 < \tau < T_0$,

$$
\begin{align*}
(h - \bar{h}, (h^2) - (\tilde{h})^2) &\in C([0,T_0]; H^3), \\
u &\in C([0,T_0]; L^q \cap D^1 \cap D^3) \cap L^2(0,T_0; D^4), \\
u_t &\in L^\infty(0,T_0; D^1) \cap L^2(0,T_0; L^q \cap D^2), \\
\sqrt{h} u_t &\in L^\infty(0,T_0; L^2), \quad \sqrt{h} u_{tt} \in L^2(0,T_0; L^2), \\
t^{1/2} u &\in L^\infty(0,T_0; D^4), \quad tu_t \in L^\infty(0,T_0; L^{q_1} \cap D^2), \\
t u_{tt} &\in L^2(0,T_0; D^1), \quad t^{1/2} \sqrt{h} u_{tt} \in L^\infty(0,T_0; L^2).
\end{align*}
$$
where

\[
\begin{cases}
q = q_1 = 2, & \text{if } \tilde{h} > 0, \\
q = 4/\alpha, \; q_1 = 8/\alpha, & \text{if } \tilde{h} = 0.
\end{cases}
\]

The aim of this paper is to obtain the global existence of classical solutions with vacuum states, our main result can be stated as follows:

**Theorem 1.2.** Assume \( \tilde{h} > 0 \) and \( \Omega = \mathbb{R}^2 \). For given numbers \( K > 0 \) (not necessarily small) and \( \bar{h} \geq \tilde{h} + 1 \), suppose that the initial data \((h_0, u_0)\) satisfy \((1.7), (1.8)\) and

\[
0 \leq \inf h_0 \leq \sup h_0 \leq \bar{h}, \quad \|u_0\|_{H^{\beta}} \leq K,
\]

for some \( 0 < \beta < 1 \). Then there exists a positive constant \( \varepsilon \) depending on \( \mu, \lambda, \tilde{h}, \) and \( K \) such that if

\[
C_0 \leq \varepsilon,
\]

the Cauchy problem \((1.2), (1.3), (1.4)\) has a unique global classical solution \((h, u)\) satisfying for any \( 0 < \tau < T < \infty \),

\[
0 \leq h(x, t) \leq 2\bar{h}, \; x \in \mathbb{R}^2, \; t \geq 0,
\]

\[
\begin{cases}
(h - \tilde{h}, (h^2 - \tilde{h}^2)) \in C([0, T]; H^3), \\
u \in C([0, T]; H^3) \cap L^2(0, T; H^4) \cap L^{\infty}(\tau, T; H^4), \\
|u_t| \in L^{\infty}(0, T; H^1) \cap L^2(0, T; H^2) \cap L^{\infty}(\tau, T; H^2) \cap H^1(\tau, T; H^1), \\
\sqrt{h}u_t \in L^{\infty}(0, T; L^2),
\end{cases}
\]

Remarks.

The structure of the paper is this, in the next section, we obtain two important lemmas, Lemma \((2.5)\) and Lemma \((2.6)\), by which we prove our main theorem 1.2 in section 3.

2. A Priori Estimates

In this section, \( \tilde{h} > 0 \), and we will establish key a priori bounds for smooth solutions to the Cauchy problem \((1.2), (1.3), (1.4)\) to extend the local classical solution guaranteed by Theorem 1.1. Thus, let \( T > 0 \) be a fixed time and \((h, u)\) be the smooth solution to \((1.2), (1.3), (1.4)\), on \( \mathbb{R}^2 \times (0, T) \) in the class \((1.9)\) with smooth initial data \((h_0, u_0)\) satisfying \((1.7), (1.8)\) and \((1.10)\).

We now state some elementary estimates which follow from Gagliardo-Nirenberg inequality and the standard \( L^p \)-estimate for the following elliptic system derived from the momentum equations in \((1.2)\):

\[
\triangle F = \text{div}(h(\dot{u} + u^\perp)), \quad \mu \triangle \omega = \nabla^\perp \cdot (h(\dot{u} + u^\perp)),
\]

for \( \nabla^\perp \triangleq (\partial_2, -\partial_1)^T \), where

\[
F \triangleq (2\mu + \lambda)\text{div} - h^2 + \tilde{h}^2, \quad \omega \triangleq \partial_2 u^1 - \partial_1 u^2,
\]

are the material derivative of \( f \), the effective viscous flux and the vorticity respectively.
Lemma 2.1. Let \((h,u)\) be a smooth solution of \((2.1)\) and \((2.2)\). For \(p \geq 2\), there exists a generic positive constant \(C\) depending only on \(\mu, \lambda\) and \(p\) such that
\[
\begin{align*}
\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} &\leq C\|h(\dot{u} + u^\perp)\|_{L^p}, \\
\|F\|_{L^p} + \|\omega\|_{L^p} &\leq C\|h(\dot{u} + u^\perp)\|^{(p-2)/p}_{L^2} (\|\nabla u\|_{L^2} + \|h^2 - \tilde{h}^2\|_{L^2})^{2/p}, \\
\|\nabla u\|_{L^p} &\leq C (\|F\|_{L^p} + \|\omega\|_{L^p}) + C\|h^2 - \tilde{h}^2\|_{L^p}.
\end{align*}
\]

Proof. The standard \(L^p\)-estimate for the elliptic system \((2.1)\) gives \((2.2)\). According to Gagliardo-Nirenberg inequality, \((2.3)\) and \((2.4)\), we have
\[
\begin{align*}
\|F\|_{L^p} + \|\omega\|_{L^p} &\leq C (\|\nabla F\|_{L^2} + \|\nabla \omega\|_{L^2})^{(p-2)/p} (\|F\|_{L^2} + \|\omega\|_{L^2})^{2/p} \\
&\leq C\|h(\dot{u} + u^\perp)\|^{(p-2)/p}_{L^2} (\|\nabla u\|_{L^2} + \|h^2 - \tilde{h}^2\|_{L^2})^{2/p}.
\end{align*}
\]
Note that \(-\Delta u = -\nabla \text{div} u - \nabla \omega\), the standard \(L^p\) estimate shows that
\[
\|\nabla u\|_{L^p} \leq C (\|\nabla \text{div} u\|_{L^p} + \|\omega\|_{L^p}), \quad \text{for } p \geq 2,
\]
which, together with \((2.5)\), gives \((2.6)\).

Lemma 2.2 (Zlotnik [22]). Let the function \(y\) satisfy
\[
y'(t) = g(y) + b'(t) \text{ on } [0,T], \quad y(0) = y^0,
\]
with \(g \in C(R)\) and \(y, b \in W^{1,1}(0,T)\). If \(g(\infty) = -\infty\) and
\[
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)
\]
for all \(0 \leq t_1 < t_2 \leq T\) with some \(N_0 \geq 0\) and \(N_1 \geq 0\), then
\[
y(t) \leq \max \{y^0, \overline{\zeta}\} + N_0 + \infty \text{ on } [0,T],
\]
where \(\overline{\zeta}\) is a constant such that
\[
g(\zeta) \leq -N_1 \quad \text{for } \zeta \geq \overline{\zeta}.
\]

Lemma 2.3 [3]. If \(\tilde{h} > 0\), then there exists some constant \(C(\tilde{h}) > 0\) such that the following estimate holds for \(h - \tilde{h} \in L^2, \psi \in D^1, h^{1/2} \psi \in L^2\),
\[
\|\psi\|_{L^2}^2 \leq C \left( \int h|\nabla \psi|^2 dx + \|h - \tilde{h}\|_{L^2}^2 \|\nabla \psi\|_{L^2}^2 \right).
\]

To estimate this solution, we set \(\sigma(t) \overset{\Delta}{=} \min\{1, t\}\) and define
\[
A_1(T) \overset{\Delta}{=} \sup_{t \in [0,T]} (\sigma \|\nabla u\|_{L^2}^2) + \int_0^T \int \sigma h \dot{u}^2 dx dt, \\
A_2(T) \overset{\Delta}{=} \sup_{t \in [0,T]} \sigma^2 \int h \dot{u}^2 dx + \int_0^T \int \sigma^2 |\nabla \dot{u}|^2 dx dt.
\]
We start with the following standard energy estimate for \((h,u)\) and the preliminary bounds for \(A_1(T)\) and \(A_2(T)\).
Lemma 2.4. Let \((p, u)\) be a smooth solution of (1.2) with \(0 \leq h(x, t) \leq \bar{h}\). Then there is a constant \(C = C(\bar{h})\) such that

\[
(2.9) \quad \sup_{0 \leq t \leq T} \int \left( \frac{1}{2} h|u|^2 + G(h) \right) dx + \int_0^T \int (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2) \, dx \, dt \leq C_0,
\]

\[
A_1(T) \leq CC_0 + C \int_{\sigma(T)}^T \int h|u|^2 \, dx \, dt + C \int_0^T \int \sigma|\nabla u|^3 \, dx \, dt, \quad (2.10)
\]

and

\[
A_2(T) \leq CC_0 + CA_1(T) + C \int_0^T \int \sigma^2|\nabla u|^4 \, dx \, dt, \quad (2.11)
\]

where \(G(h)\) is given by (1.3).

**Proof.** Multiplying the \((1.2)_1\) by \(G'(h)\) and \((1.2)_2\) by \(u\) and integrating, one shows directly the energy inequality (2.10).

The proof of \((1.2)_1\) and \((1.2)_2\) follows the idea in Hoff (1990). For integer \(m \geq 0\), multiplying \((1.2)_2\) by \(\sigma^m \dot{u}\) then integrating the resulting equality over \(\mathbb{R}^2\) leads to

\[
\int \sigma^m h|\dot{u}|^2 \, dx = \int (-\sigma^m \dot{u} \cdot \nabla (h^2) + \mu \sigma^m \Delta u \cdot \dot{u} + (\lambda + \mu)\sigma^m \nabla \cdot (\sigma^m \dot{u}) - \sigma^m hu^0 \cdot \dot{u}) \, dx
\]

\[
= \sum_{i=1}^4 M_i.
\]

Using \((1.2)_1\) and integrating by parts give

\[
M_1 = -\int \sigma^m \dot{u} \cdot \nabla (h^2) \, dx
\]

\[
= \int (\sigma^m (\text{div} u)\dot{h}^2 - \bar{h}^2) - \sigma^m (u \cdot \nabla u) \cdot \nabla (h^2) \, dx
\]

\[
= \left( \int \sigma^m \text{div}(u^2 - \bar{h}^2) \, dx \right)_t - m \sigma^m \sigma' \int \text{div}(u^2 - \bar{h}^2) \, dx
\]

\[
+ \int \sigma^m (h^2(\text{div} u)^2 + (\bar{h}^2) \partial_i u \cdot \partial_j u) \, dx
\]

\[
\leq \left( \int \sigma^m \text{div}(u^2 - \bar{h}^2) \, dx \right)_t + C(\bar{h})\|\nabla u\|_{L^2}^2 + C(\bar{h})m^2 \sigma m \sigma' C_0.
\]

Integration by parts implies

\[
M_2 = \int \mu \sigma^m \Delta u \cdot \dot{u} \, dx
\]

\[
= -\frac{\mu}{2} \left( \sigma^m \|\nabla u\|_{L^2}^2 \right)_t + \frac{\mu m}{2} \sigma^m \|\nabla u\|_{L^2}^2 - \mu \sigma^m \int \partial_i u \partial_j (u^k \partial_k u^j) \, dx
\]

\[
\leq -\frac{\mu}{2} \left( \sigma^m \|\nabla u\|_{L^2}^2 \right)_t + Cm \sigma^{m-1} \|\nabla u\|_{L^2}^2 + C \int \sigma^m |\nabla u|^3 \, dx,
\]

and similarly,

\[
M_3 \leq -\frac{\lambda + \mu}{2} \left( \sigma^m \|\text{div} u\|_{L^2}^2 \right)_t + Cm \sigma^{m-1} \|\nabla u\|_{L^2}^2 + C \int \sigma^m |\nabla u|^3 \, dx,
\]

Moreover,

\[
M_4 = -\int \sigma^m hu^0 \cdot \dot{u} \, dx \leq 1/2 \int \sigma^m h|\dot{u}|^2 \, dx + 1/2 \int \sigma^m h|u|^2 \, dx.
\]
Similarly, integration by parts leads to
\[ B'(t) + 1/2 \int \sigma^m h |\dot{u}|^2 dx \]
\[ \leq (C m \sigma^{m-1} + C(\bar{h})) \| \nabla u \|_{L^2}^2 + C \int \sigma^m h |u|^2 dx \]
\[ + C(\bar{h}) m^2 \sigma^{2(m-1)} \sigma' C_0 + C \int \sigma^m |\nabla u|^3 dx, \]  
(2.17)  
where
\[ B(t) \triangleq \frac{\mu \sigma^m}{2} \| \nabla u \|_{L^2}^2 + \frac{(\lambda + \mu) \sigma^m}{2} \| \text{div} u \|_{L^2}^2 - \int \sigma^m \text{div}(h^2 - \bar{h}^2) dx \]
\[ \geq \frac{\mu \sigma^m}{2} \| \nabla u \|_{L^2}^2 + \frac{(\lambda + \mu) \sigma^m}{2} \| \text{div} u \|_{L^2}^2 - C \sigma^m C_0^{1/2} \| \text{div} u \|_{L^2} \]  
(2.18)  
\[ \geq \frac{\mu \sigma^m}{4} \| \nabla u \|_{L^2}^2 + \frac{(\lambda + \mu) \sigma^m}{2} \| \text{div} u \|_{L^2}^2 - C \sigma^m C_0. \]

Integrating (2.17) over (0, T), choosing \( m = 1 \), and using (2.18), one gets (2.19).

Next, for integer \( m \geq 0 \), operating \( \sigma^m \dot{u}^j (\partial_j \partial_t + \text{div}(u^j)) \) to (2.19), summing with respect to \( j \), and integrating the resulting equation over \( \mathbb{R}^2 \), one obtains after integration by parts
\[ \left( \frac{\sigma^m}{2} \int h |\dot{u}|^2 dx \right)_t - \frac{m}{2} \sigma^{m-1} \sigma' \int h |\dot{u}|^2 dx \]
\[ = - \int \sigma^m \dot{u}^j (\partial_j (h^2) + \partial_i \text{div} u \partial_j h^2 u) dx + \mu \int \sigma^m \dot{u}^j [\Delta u^j + \text{div}(u \Delta u^j)] dx \]
\[ + (\lambda + \mu) \int \sigma^m \dot{u}^j [\partial_i \partial_j \text{div} u + \text{div}(u \partial_j \text{div} u)] dx \]
\[ \triangleq \sum_{i=1}^{3} N_i. \]

It follows from integration by parts and using the equation (2.19) that
\[ N_1 = - \int \sigma^m \dot{u}^j (\partial_j (h^2) + \partial_i \text{div} u \partial_j h^2 u) dx \]
\[ = \int \sigma^m [-2h^2 \text{div} u \partial_j \dot{u}^j + \partial_k (\partial_j \dot{w}_k u^k) h^2 - h^2 \partial_j (\partial_k \dot{w}_k u^k)] dx \]
\[ \leq \delta \sigma^m \| \nabla \dot{u} \|_{L^2}^2 + C(\bar{h}, \delta) \sigma^m \| \nabla u \|_{L^2}^2. \]  
(2.20)  
Integration by parts leads to
\[ N_2 = \mu \int \sigma^m \dot{u}^j [\Delta u^j + \text{div}(u \Delta u^j)] dx \]
\[ = \mu \int \sigma^m [\| \nabla \dot{u} \|^2 + \partial_i \dot{u}^j \partial_k u^k \partial_i u^k - \partial_i \dot{u}^j \partial_k u^k \partial_k \dot{u}^j - \partial_i u^j \partial_i \partial_j \dot{w}_k] dx \]
\[ \leq \frac{-3 \mu}{4} \int \sigma^m |\nabla \dot{u}|^2 dx + C \int \sigma^m |\nabla u|^3 dx. \]  
(2.21)  
Similarly,
\[ N_3 \leq -\frac{\mu + \lambda}{2} \int \sigma^m (\text{div} u)^2 dx + C \int \sigma^m |\nabla u|^4 dx. \]  
(2.22)
Substituting (2.15), (2.17) into (2.13) shows that for \( \delta \) suitably small, it holds that
\[
\left( \sigma^m \int h|\dot{u}|^2dx \right)_t + \mu \int \sigma^m|\nabla \dot{u}|^2dx + (\mu + \lambda) \int \sigma^m(\text{div} \dot{u})^2dx \\
\leq m\sigma^{m-1}\sigma' \int h|\dot{u}|^2dx + C\sigma^m\|\nabla u\|_{L^4}^4 + C(\bar{h})\sigma^m\|\nabla u\|_{L^2}^2.
\] (2.23)

Taking \( m = 2 \) in (2.23) and noticing that
\[
2 \int_0^T \sigma^2 \int h\dot{u}^2dxdt \leq CA_1(T),
\]
we immediately obtain (2.20) after integrating (2.23) over \((0, T)\).

We have the following key a priori estimates on \((h, u)\).

First, the following lemma is motivated by the idea of Hoff \cite{Hoff} and H-L-Xin \cite{HLX}.

**Lemma 2.5.** For given numbers \( K > 0 \) and \( \bar{h} \geq \bar{h} + 1 \), assume that \((h_0, u_0)\) satisfy (1.2) and (1.3). Then there exist a positive constant \( \varepsilon \) depending on \( \mu, \lambda, \bar{h}, \bar{h} \) and \( K \) such that if \((h, u)\) is a smooth solution of (1.2) on \( \mathbb{R}^2 \times (0, T) \) satisfying
\[
\begin{cases}
\sup_{\mathbb{R}^2 \times [0, T]} h \leq 2\bar{h}, \\
A_1(T) + A_2(T) \leq 2C_0^{1/2},
\end{cases}
\] (2.24)

the following estimates hold
\[
\sup_{\mathbb{R}^2 \times [0, T]} h \leq \frac{7}{4}\bar{h}, \quad A_1(T) + A_2(T) \leq C_0^{1/2},
\] (2.25)
provided \( C_0 \leq \varepsilon \).

**Proof.** Step 1 In this step, we derive the basic estimates on velocity field and its material derivatives, i.e. \( A_1(T) + A_2(T) \). Lemma 2.4 shows that
\[
A_1(T) + A_2(T) \leq C(\bar{h})C_0 + C \int_{\sigma(T)}^T \int h|u|^2dxdt \\
+ C(\bar{h}) \int_0^T \sigma^2\|\nabla u\|_{L^4}^4ds + C(\bar{h}) \int_0^T \sigma\|\nabla u\|_{L^2}^2ds.
\] (2.26)

Due to (2.20),
\[
\int_0^T \sigma^2\|\nabla u\|_{L^4}^4ds \leq C \int_0^T \sigma^2(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4)ds + C \int_0^T \sigma^2|h^2 - \bar{h}^2|_{L^4}^4ds.
\] (2.27)

It follows from (2.20) that
\[
\int_0^T \sigma^2(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4)ds \\
\leq C(\bar{h}) \sup_{t \in (0, T]} (\sigma\|\nabla u\|_{L^2}^2 + C_0) \int_0^T \sigma(h|u|^2 + |h|u|^2)dxds \\
\leq C(\bar{h}) (A_1(T) + C_0) \left( A_1(T) + C_0 + \int_{\sigma(T)}^T \int h|u|^2dxds \right) \\
\leq C(\bar{h})C_0 + CC_0^{1/2} \int_{\sigma(T)}^T \int h|u|^2dxds.
\] (2.28)
To estimate the second term on the right hand side of (2.27), one deduces from (1.2) that \( h^2 - \tilde{h}^2 \) satisfies

\[
(h^2 - \tilde{h}^2)_t + u \cdot \nabla (h^2 - \tilde{h}^2) + 2(h^2 - \tilde{h}^2) \text{div} u + 2\tilde{h}^2 \text{div} u = 0.
\]

(2.29)

Multiplying (2.29) by \( 3(h^2 - \tilde{h}^2)^2 \) and integrating the resulting equality over \( \mathbb{R}^2 \), one gets after using (2.2) that

\[
\frac{5}{2\mu + \lambda} \| h^2 - \tilde{h}^2 \|_{L^4}^4
= - \left( \int (h^2 - \tilde{h}^2)^3 dx \right)_t - \frac{5}{2\mu + \lambda} \int (h^2 - \tilde{h}^2)^3 F dx
- 6\tilde{h}^2 \int (h^2 - \tilde{h}^2)^2 \text{div} u dx
\leq - \left( \int (h^2 - \tilde{h}^2)^3 dx \right)_t + \delta \| h^2 - \tilde{h}^2 \|_{L^4}^4 + C_0 \| F \|_{L^4}^4 + C_\delta \| \nabla u \|_{L^4}^2.
\]

(2.30)

Multiplying (2.30) by \( \sigma^2 \), integrating the resulting inequality over \((0, T)\), and choosing \( \delta \) suitably small, one may arrive at

\[
\int_0^T \sigma^2 \| h^2 - \tilde{h}^2 \|_{L^4}^4 dt
\leq C \sup_{0 \leq t \leq T} \| h^2 - \tilde{h}^2 \|_{L^4}^3 + C \int_0^{\sigma(T)} \| h^2 - \tilde{h}^2 \|_{L^4}^3 dt
+ C(\tilde{h}) \int_0^T \sigma^2 \| F \|_{L^4}^4 dt + C(\tilde{h}) C_0
\leq C(\tilde{h}) C_0 + C C_0^{1/2} \int_{\sigma(T)}^T \int h |u|^2 dx dt,
\]

(2.31)

where (2.28) has been used. Therefore, collecting (2.27), (2.29) and (2.31) shows that

\[
\int_0^T \sigma^2 \left( \| \nabla u \|_{L^4}^4 + \| h^2 - \tilde{h}^2 \|_{L^4}^4 \right) dt \leq C(\tilde{h}) C_0 + C C_0^{1/2} \int_{\sigma(T)}^T \int h |u|^2 dx dt.
\]

(2.32)

Finally, we estimate the last term on the right hand side of (2.28). First, (2.32) implies that

\[
\int_{\sigma(T)}^T \int \sigma |\nabla u|^3 dx ds \leq \int_{\sigma(T)}^T \int (|\nabla u|^4 + |\nabla u|^2) dx ds
\leq C C_0 + C C_0^{1/2} \int_{\sigma(T)}^T \int h |u|^2 dx ds.
\]

(2.33)
Next, one deduces from (2.30) and (2.34) that
\[
\int_0^{\sigma(T)} \sigma \| \nabla u \|^2_{L^2} ds 
\leq C \int_0^{\sigma(T)} \sigma \left( \| F \|^2_{L^2} + \| \omega \|^2_{L^2} + \| h^2 - \tilde{h}^2 \|^2_{L^2} \right) ds
\leq C \int_0^{\sigma(T)} \sigma \left( \| F \|^2_{L^2} + \| \omega \|^2_{L^2} \| h \dot{u} + hu^\perp \|_{L^2} + \| h^2 - \tilde{h}^2 \|^2_{L^2} \right) ds
\leq C \delta^{-1} \int_0^{\sigma(T)} \sigma \left( \| \nabla u \|^2_{L^2} + \| h^2 - \tilde{h}^2 \|^2_{L^2} \right) ds + \delta \int_0^{\sigma(T)} \sigma \| h \dot{u} \|^2_{L^2} ds + CC_0
\leq (C \delta^{-1} C_0 + \delta) A_1(\sigma(T)) + CC_0
\leq C(\tilde{h}) C_0.
\]

According to (2.24), (2.25), (2.27),(2.34), we have
\[
A_1(T) + A_2(T) \leq CC_0 + CC_0 \int_{\sigma(T)}^{T} \| h^{1/2} u \|_{L^2} dt.
\]

Due to the decay properties of solutions (see (2.3), (2.17)), \( \int_{\sigma(T)}^{T} \| h^{1/2} u \|_{L^2} dt \) is bounded by a constant \( C \) independent of \( T \). Therefore,
\[
A_1(T) + A_2(T) \leq C_0^{1/2},
\]
provided \( C_0 \leq \varepsilon_0 \), where \( \varepsilon_0 \) depends on \( \mu, \lambda, \tilde{h}, \bar{h}, C \).

**Step2** In this step, we deal with the short time energy estimates. In fact, we claim that for \( K \) and \( \beta \) as in (2.37),
\[
\sup_{0 \leq t \leq \sigma(T)} t^{1-\beta} \int |\nabla u(\cdot, t)|^2 dx \leq C(\bar{\eta}, K).
\]  

The proof is similar as in (2.37). Fixing the local-in-time solution \((h, u)\), we decompose \( u = w_1 + w_2 \), where \( w_1 \) and \( w_2 \) are defined by
\[
L w_1 = 0, \quad w_1(x, 0) = u_0(x),
\]  

and
\[
L w_2 = -\nabla P(\rho) - hu^\perp, \quad w_2(x, 0) = 0,
\]  

respectively, with \( L \) the differential operator acting on functions \( w : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^2 \) defined by
\[
(Lw)^j = hw^j_1 + hu \cdot \nabla w^j - (\mu \Delta w^j + (\mu + \lambda) \text{div} w_x^j).
\]

Straightforward energy estimate shows that:
\[
\sup_{t \geq 0} \int h |w_1|^2 dx + \int_0^\infty \int |\nabla w_1|^2 dx dt \leq C(h) \int |u_0|^2 dx.
\]  

Multiplying (2.30) by \( \sigma^m \dot{w}_1 \), integrating the resulting equality over \( \mathbb{R}^2 \), we get
\[
\frac{d}{dt} \left( \sigma^m \int |\nabla w_1|^2 dx \right) + \int \sigma^m h|w_1|^2 dx \leq C(m\sigma^m - 1 \sigma' + \sigma^m \| \nabla u \|_{L^2}) \| \nabla w_1 \|_{L^2}^2,
\]
which together with \( a_{16} \) and Gronwall’s inequality gives

\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla w_1 \|_{L^2}^2 \leq e^{CC_0} \| \nabla u_0 \|_{L^2}^2, \quad \sup_{0 \leq t \leq \sigma(T)} t \| \nabla w_1 \|_{L^2}^2 \leq C(\tilde{h}) \| u_0 \|_{L^2}^2.
\]

Then use the K-method interpolation (see Chapter 22), we have for any \( t \in (0, \sigma(T)) \),

\[
\| \nabla w_1 \|_{(L^2, L^2)_{(1-\beta), 2}} \leq \left( e^{CC_0} \right)^{1-\beta} \left( C(\tilde{h}) \right)^{1/2} \| u_0 \|_{(H^1, H^0)_{(1-\beta), 2}}.
\]

Note that \((L^2, L^2)_{1-\beta, 2} = L^2 \) and \((H^1, L^2)_{1-\beta, 2} = H^s\) (see Chapter 32 in [20]). Hence, we have

\[
\sup_{0 \leq t \leq \sigma(T)} t^{1-\beta} \| \nabla w_1 \|_{L^2}^2 \leq C(\tilde{h}) \| u_0 \|_{H^\beta}^2.
\]

(2.39)  

Next, multiplying \((2.37)\) by \( w_2 \) and integrating the resulting equality over \( \mathbb{R}^2 \) lead to

\[
\frac{d}{dt} \left( \int |\nabla w_2|^2 dx \right) + \int h |w_2|^2 dx 
\leq \frac{d}{dt} \left( \int (h^2 - \tilde{h}^2) |w_2|^2 dx \right) + C \| \nabla u \|_{L^2}^2 \| \nabla w_2 \|_{L^2}^2 + C_0 \| \nabla u \|_{L^2}^2 + C_0^2 \| \nabla w_2 \|_{L^2}^2 + C(\tilde{h}) C_0,
\]

which gives

\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla w_2 \|_{L^2} \leq C(\tilde{h}),
\]

(2.40)  

due to \( w_2(x, 0) = 0 \). Combining \((2.20)\) with \((2.40)\) yields \((2.41)\).

**Step3** In this step, we estimate the effective viscous flux \( F \) defined by \((2.20)\). For \( r = 6/\beta \), we get

\[
\int_0^{\sigma(T)} \| F(\cdot, t) \|_{L^\infty} dt 
\leq C \int_0^{\sigma(T)} \| F(\cdot, t) \|_{L^2}^{r-2} \| \nabla F(\cdot, t) \|_{L^2}^r dt
\leq C \int_0^{\sigma(T)} \left( \| \nabla u \|_{L^2} + C_0^{1/2} \right)^{r-2} \left( \| h^{1/2} \dot{u} \|_{L^2} + \| \nabla \dot{u} \|_{L^2} + \| h^{1/2} u \|_{L^2} + \| \nabla u \|_{L^2} \right)^{2(r-1)} dt
\leq C \int_0^{\sigma(T)} \left( \sigma^{-1/2} \| \nabla u \|_{L^2} \right)^{r-2} \left( \sigma \| h^{1/2} \dot{u} \|_{L^2} + \sigma \| \nabla \dot{u} \|_{L^2} + \sigma C_0^{1/2} \right)^{2(r-1)} \sigma^{- \frac{(3-\beta)(3-2\beta)}{4(3-1)}} dt
\]

\[
+ C C_0^{\frac{r-2}{4(r-1)}} \int_0^{\sigma(T)} \left( \sigma \| h^{1/2} \dot{u} \|_{L^2} + \sigma \| \nabla \dot{u} \|_{L^2} + \sigma C_0^{1/2} \right)^{2(r-1)} \sigma^{- \frac{3r}{4(r-1)}} dt
\]

\[
\leq C K \left( A_2(\sigma(T)) + C_0 \right)^{\frac{r-2}{4(r-1)}} \left( \int_0^{\sigma(T)} \sigma^{- \frac{(3-\beta)(3-2\beta)}{3r-4}} dt \right)^{\frac{3r-4}{4(r-1)}}
\]

\[
+ C C_0^{\frac{r-2}{4(r-1)}} \left( A_2(\sigma(T)) + C_0 \right)^{\frac{r}{4(r-1)}} \left( \int_0^{\sigma(T)} \sigma^{- \frac{2r}{3r-4}} dt \right)^{\frac{3r-4}{4(r-1)}}
\]

\[
\leq C(K) C_0^{1/4},
\]

(2.41)  

where we have used \( \max \left\{ \frac{r(3-\beta) - 2(1-\beta)}{3r-4}, \frac{2r}{3r-4} \right\} < 1 \), since \( r = 6/\beta \).
On the other hand, for all \( \sigma(T) \leq t_1 \leq t_2 \leq T \),
\[
\int_{t_1}^{t_2} \| F(\cdot, t) \|_{L^\infty}^2 \, dt \leq C \int_{t_1}^{t_2} \| F(\cdot, t) \|_{L^2} \| \nabla F(\cdot, t) \|_{L^1}^2 \, dt
\]
\[
\leq C(h) C_0^{1/4} \int_{\sigma(T)}^{T} \left( \| h^{1/2} u \|_{L^2}^2 + \| \nabla u(\cdot, t) \|_{L^2}^2 + \| \nabla u(\cdot, t) \|_{L^2}^2 \right) \, dt
\]
\[
+ C(h) C_0^{1/4} \int_{t_1}^{t_2} \| h^{1/2} u \|_{L^2}^2 \, dt
\]
\[
\leq C(h) C_0^{3/4} \left( 1 + C_0^{1/2} (t_2 - t_1) \right).
\]

Step 4 In this step, we do the super-norm estimates on the density. Rewrite the continuity equation \((2.2)_1\) as
\[
D_t h = g(h) + b'(t),
\]
where
\[
D_t h \equiv h_t + u \cdot \nabla h, \quad g(h) \equiv -\frac{ah}{2\mu + \lambda} (h^2 - \tilde{h}^2), \quad b(t) \equiv -\frac{1}{2\mu + \lambda} \int_0^t hF dt.
\]
Next, for all \( 0 \leq t_1 < t_2 \leq \sigma(T) \) and \( r = 6/\beta \), by \( \text{(2.1)} \), we have
\[
|b(t_2) - b(t_1)| \leq C \int_0^{\sigma(T)} \| (hF)(\cdot, t) \|_{L^\infty} \, dt \leq C(h, K) C_0^{1/4}.
\]
Therefore, one can choose \( N_0 \) and \( N_1 \) in \((2.10)\) as follows:
\[
N_1 = 0, \quad N_0 = C(h, K) C_0^{1/4},
\]
and \( \tilde{\zeta} = \tilde{h} \) in \((2.7)\). Then
\[
g(\zeta) = -\frac{a\tilde{h}}{2\mu + \lambda} (\zeta^2 - \tilde{h}^2) \leq -N_1 = 0, \quad \text{for all} \quad \zeta \geq \tilde{\zeta} = \tilde{h}.
\]
Lemma \((2.2)\) thus yields that
\[
\sup_{t \in [0, \sigma(T)]} \| h \|_{L^\infty} \leq \max \{ \bar{h}, \tilde{h} \} + N_0 \leq \bar{h} + C(h, K) C_0^{1/4} \leq \frac{3\bar{h}}{2}, \quad \text{(2.42) \ \ a103}
\]
provided
\[
C_0 \leq \varepsilon_1, \quad \text{for} \ \varepsilon_1 \equiv \min \left\{ \varepsilon_0, \left( \frac{\bar{h}}{2C(h, K)} \right)^4 \right\}.
\]
On the other hand, for all \( \sigma(T) \leq t_1 \leq t_2 \leq T \),
\[
|b(t_2) - b(t_1)| \leq C(h) \int_{t_1}^{t_2} \| F(\cdot, t) \|_{L^\infty} \, dt
\]
\[
\leq \frac{a}{2\mu + \lambda} (t_2 - t_1) + C(h) \int_{t_1}^{t_2} \| F(\cdot, t) \|_{L^\infty}^2 \, dt
\]
\[
\leq \left( \frac{a}{2\mu + \lambda} + C(h) C_0 \right) (t_2 - t_1) + C(h) C_0^{3/4}.
\]
One can choose \( N_1 \) and \( N_0 \) in \((2.10)\) as:
\[
N_1 = \frac{a}{2\mu + \lambda} + C_0 C(h), \quad N_0 = C(h) C_0^{3/4}.
\]
Since $\tilde{h} > 0$, we have
\[
g(\zeta) = -\frac{\zeta}{2\mu + \lambda}(\zeta^2 - \tilde{h}^2) < -\frac{1}{2\mu + \lambda}, \quad \text{for all } \zeta \geq \tilde{h} + 1.
\]
Therefore,
\[
g(\zeta) \leq -N_1 = -\frac{1}{2\mu + \lambda} - C_0 C(\tilde{h}), \quad \text{for all } \zeta \geq \tilde{h} + 1,
\]
provided
\[
C_0 \leq \varepsilon_2 \triangleq \min\{\varepsilon_1, \frac{2\tilde{h}^2 + 3\tilde{h}}{(2\mu + \lambda)C(\tilde{h})}\}.
\]
So one can set $\bar{\zeta} = \tilde{h} + 1$ in (2.7). Lemma 2.2 and (2.41) thus yield that
\[
\sup_{t \in \sigma(T), T} \|h\|_{L^\infty} \leq \max\left\{\frac{3\tilde{h}}{2}, \tilde{h} + 1\right\} + N_0 \leq \frac{3\tilde{h}}{2} + C(\tilde{h})C_0^{3/4} \leq \frac{7\tilde{h}}{4}, \tag{2.43}
\]
provided
\[
C_0 \leq \varepsilon \triangleq \min\left\{\varepsilon_2, \left(\frac{\tilde{h}}{4C(\tilde{h})}\right)^{4/3}\right\}.
\]
The combination of (2.41) with (2.42) completes the proof of the uniform upper bounds for density in (2.25).

**Lemma 2.6.** Suppose that $(h_0, u_0)$ satisfy the assumptions in Lemma 2.5 and $(h, u)$ is a smooth solution of (1.2) (1.3) (1.4) on $\mathbb{R}^2 \times (0, T]$. Then the velocity lies in the following class:
\[
\nabla u_t, \nabla^3 u \in C([\tau, T]; L^p), \quad \nabla u, \nabla^2 u \in C([\tau, T]; L^2 \cap C(\mathbb{R}^2)) \tag{2.44}
\]
for all $p \geq 2$.

**Proof.** We assume the constant $C$ may depend on $T$, $\|g\|_{L^2}$, $\|h_0^{1/2}g\|_{L^2}$ and the initial data.

**Step 1** Taking $m = 0$ in (2.23), one deduce from Gagliardo-Nirenberg’s inequality and (2.40), that
\[
\left(\int h|\tilde{u}|^2dx\right)_t + \mu \int |\nabla \tilde{u}|^2dx + (\mu + \lambda) \int (\text{div}\tilde{u})^2dx \\
\leq C\|\nabla u\|_{L^4}^4 + C(\tilde{h})\|\nabla u\|_{L^2}^2 \\
\leq C\|F\|_{L^4}^4 + C\|\omega\|_{L^4}^4 + C \\
\leq C\left(\|F\|_{L^2}^2 + \|\omega\|_{L^2}^2\right)\left(\|h\tilde{u}\|_{L^2}^2 + \|hu\|_{L^2}^2\right) + C \\
\leq C\|h\tilde{u}\|_{L^2}^2 + C.
\]

Taking into account on the compatibility condition (1.9), we can define
\[
\sqrt{h\tilde{u}}(x, t = 0) = \sqrt{h_0}g.
\]
Then Gronwall’s inequality gives
\[
\sup_{0 \leq t \leq T} \int h|\tilde{u}|^2dx + \int_0^T \int |\nabla \tilde{u}|^2dxdt \leq C. \tag{2.45}
\]
Next we claim the estimates on the spatial gradient of the solution \((h, u)\) as following:

\[
\sup_{0 \leq t \leq T} (\|\nabla h\|_{L^2} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla u\|_{L^\infty} dt \leq C(q) \tag{2.46}
\]

for any \(q > 2\). Then according to (2.45) and (2.46), we have

\[
\sup_{0 \leq t \leq T} \|u\|_{L^2 \cap L^\infty} \leq C,
\]

which together with (2.47) and the definition of \(\dot{u}\) gives

\[
\sup_{0 \leq t \leq T} \int h|u_t|^2 dx + \int_0^T \|\nabla u_t\|^2 dt \leq C. \tag{2.47}
\]

**Step 2** In this step we will deal with some high order estimates of the solutions. Note that \(h^2\) satisfies

\[
(h^2)_t + u \cdot \nabla(h^2) + 2h^2 \text{div} u = 0, \tag{2.48}
\]

and

\[
\|\nabla u\|_{H^m} \leq C \left( \|\text{div} u\|_{H^m} + \|\omega\|_{H^m} \right) \\
\quad \leq C \left( \|F\|_{H^m} + \|\omega\|_{H^m} + \|h^2 - \tilde{h}^2\|_{H^m} \right), \quad \text{for} \ m = 1, 2.
\]

Therefore,

\[
\frac{d}{dt} \left( \|\nabla^2(h^2)\|^2_{L^2} + \|\nabla^2 h\|^2_{L^2} \right) \\
\quad \leq C(1 + \|\nabla u\|_{L^\infty}) \left( \|\nabla^2(h^2)\|^2_{L^2} + \|\nabla^2 h\|^2_{L^2} \right) + C\|F\|^2_{H^2} \tag{2.49}
\]

According to the definition of \(F\) and \(\omega\), we get by the standard \(L^2\)-estimate for elliptic system, (2.15) and (2.16) that

\[
\|F\|_{H^2} + \|\omega\|_{H^2} \leq C \left( \|F\|_{L^2} + \|u\|_{L^2} + \|h(u + u^\perp)\|_{L^2} + \|\nabla(h^\perp + hu^\perp)\|_{L^2} \right) \\
\quad \leq C(1 + \|\nabla h\|_{L^4} \|u\|_{L^4} + \|\nabla u\|_{L^2} + \|\nabla u^\perp\|_{L^2}) \\
\quad \leq C(1 + \|\nabla u\|_{L^2}),
\]

which, together with (2.47) and Gronwall’s inequality, gives directly

\[
\sup_{t \in [0, T]} \left( \|\nabla^2(h^2)\|^2_{L^2} + \|\nabla^2 h\|^2_{L^2} \right) \leq C.
\]

Thus

\[
\sup_{t \in [0, T]} (\|h - \tilde{h}\|_{H^2} + \|h^2 - \tilde{h}^2\|_{H^2}) \leq C. \tag{2.50}
\]

Moreover, one deduces from (2.48) and (2.49) that

\[
\|(h^2)_t\|_{L^2} \leq C\|u\|_{L^\infty}\|\nabla(h^2)\|_{L^2} + C\|\nabla u\|_{L^2} \leq C. \tag{2.51}
\]

Differentiating (2.48) yields

\[
\nabla(h^2)_t + u \cdot \nabla^2(h^2) + \nabla u \cdot \nabla(h^2) + 2\nabla(h^2)\text{div} u + 2h^2\nabla\text{div} u = 0.
\]
In this step, we will derive the higher order estimates of the solutions which are needed to guarantee the lemma. Thus, the combination of (2.50) with (2.52) implies
\[
\sup_{0 \leq t \leq T} \| (h^2)_t \|_{H^1} \leq C.
\]
Note that \((h^2)_t\) satisfies
\[
(h^2)_t + 2(h^2)_t \text{div} u + 2h^2 \text{div} u_t + u_t \cdot \nabla (h^2) + u \cdot \nabla (h^2)_t = 0.
\]
Thus,
\[
\int_0^T \| (h^2)_t \|_{L^2}^2 \, dt \\
\leq C \int_0^T \left( \| (h^2)_t \|_{L^4} \| \nabla u \|_{L^4} + \| \nabla u_t \|_{L^2} + \| u_t \|_{L^4} \| \nabla (h^2) \|_{L^4} + \| \nabla (h^2)_t \|_{L^2} \right)^2 \, dt \\
\leq C.
\]
One can handle \(h_t\) and \(h_{tt}\) similarly. Thus
\[
\sup_{0 \leq t \leq T} \left( \| h_t \|_{H^1} + \| (h^2)_t \|_{H^1} \right) + \int_0^T \left( \| h_{tt} \|_{L^2}^2 + \| (h^2)_{tt} \|_{L^2}^2 \right) \, dt \leq C. \tag{2.53}
\]

Step 3 In this step, we will derive the higher order estimates of the solutions which are needed to guarantee the lemma.

First, we claim the following estimates:
\[
\sup_{0 \leq t \leq T} \| u_t \|_{H^1} + \int_0^T \int \| h_{tt} \|^2 \, dx \, dt \leq C, \tag{2.54}
\]
\[
\sup_{\tau \leq t \leq T} \int h_{tt} \| u_t \|^2 \, dx + \int_\tau^T \int \| \nabla u_t \|^2 \, dx \, dt \leq C(\tau) \quad \text{for any } \tau \in (0, T).
\]

Next, it follows from (2.50) and (2.61) that
\[
\| \nabla (h\dot{u}) \|_{L^2} \leq \| \nabla h \|_{L^3} \| u_t \|_{L^6} + C \| \nabla u_t \|_{L^2} + C \| \nabla h \|_{L^3} \| u \|_{L^\infty} \| \nabla u \|_{L^6} \\
+ C \| \nabla u \|_{L^3} \| \nabla u_t \|_{L^6} + C \| u \|_{L^\infty} \| \nabla^2 u \|_{L^2} \\
\leq C,
\]
which together with (2.46) gives
\[
\sup_{0 \leq t \leq T} \| h\dot{u} \|_{H^1} \leq C. \tag{2.55}
\]

The standard \(H^1\)-estimate for elliptic system gives
\[
\| \nabla^2 u \|_{H^1} \leq C \| \mu \Delta u + (\mu + \lambda) \nabla \text{div} u \|_{H^1} \\
= C \| h\dot{u} + h u + \nabla (h^2) \|_{H^1} \\
\leq C,
\]
due to (2.2), (2.3) and (2.9). As a consequence of (2.50) and (2.54), one has
\[
\sup_{0 \leq t \leq T} \| \nabla u \|_{H^2} \leq C. \tag{2.56}
\]
Therefore, the standard $L^2$-estimate for elliptic, \(\text{sp23}\), and \(\text{sp24}\) yield that
\[
\|\nabla^2 u_t\|_{L^2} \leq C\|\mu \Delta u_t + (\mu + \lambda) \nabla \text{div} u_t\|_{L^2} \\
\leq C \left( \|hu_t\|_{L^2} + \|h_t\|_{L^3} \|u_t\|_{L^6} + \|h_t\|_{L^3} \|u_t\|_{L^\infty} \|\nabla u_t\|_{L^6} + 1 \right) \\
+ C \left( \|u_t\|_{L^6} \|\nabla u_t\|_{L^2} + \|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} + \|\nabla (h^2)_t\|_{L^2} \right) \\
\leq C \|hu_t\|_{L^2} + C,
\]
which, together with \(\text{sp27}\), implies
\[
\int_0^T \|\nabla u_t\|^2_{H^1} \, dt \leq C. \tag{2.58} \text{sp24}
\]

Applying the standard $H^2$-estimate for elliptic system again and \(\text{sp38}\) lead to
\[
\|\nabla^2 u\|_{H^2} \leq C\|\mu \Delta u + (\mu + \lambda) \nabla \text{div} u\|_{H^2} \\
\leq C \|h\dot{u}\|_{H^2} + C \|h u^{-1}\|_{H^2} + C \|\nabla(h^2)\|_{H^2} \tag{2.59} \text{sp38}
\]
where one has used \(\text{sp19}\) and the following estimates:
\[
\|\nabla^2 (h u_t)\|_{L^2} \leq C \left( \|\nabla^2 h\|_{L^2} \|u_t\|_{L^2} + \|\nabla h\| \|\nabla u_t\|_{L^2} + \|\nabla^2 u_t\|_{L^2} \right) \\
\leq C \left( \|\nabla^2 h\|_{L^2} \|u_t\|_{H^1} + \|\nabla h\|_{L^3} \|u_t\|_{L^6} + \|\nabla^2 u_t\|_{L^2} \right) \\
\leq C + C \|\nabla u_t\|_{H^1},
\]
and
\[
\|\nabla^2 (hu \cdot \nabla u)\|_{L^2} \leq C \left( \|\nabla^2 (hu)\| \|\nabla u\|_{L^2} + \|\nabla (hu)\| \|\nabla^2 u\|_{L^2} + \|\nabla^2 u\|_{L^2} \right) \\
\leq C \left( 1 + \|\nabla^2 (hu)\|_{L^2} \|\nabla u\|_{H^1} + \|\nabla (hu)\|_{L^3} \|\nabla^2 u\|_{L^6} \right) \\
\leq C \left( 1 + \|\nabla^2 h\|_{L^2} \|u\|_{L^\infty} + \|\nabla h\|_{L^6} \|\nabla u\|_{L^3} + \|\nabla^2 u\|_{L^2} \right) \\
\leq C,
\]
due to \(\text{sp21}\) and \(\text{sp20}\). By using \(\text{sp21}\), \(\text{sp38}\), and \(\text{sp19}\), one may get that
\[
\left( \|\nabla^3 (h^2)\|_{L^2} \right)_t \\
\leq C \left( \|\nabla^3 u\| \|\nabla^2 (h^2)\|_{L^2} + \|\nabla^2 u\| \|\nabla^2 (h^2)\|_{L^2} + \|\nabla u\| \|\nabla^3 (h^2)\|_{L^2} + \|\nabla^3 u\|_{L^2} \right) \|\nabla^3 (h^2)\|_{L^2} \\
\leq C \left( \|\nabla^3 u\|_{L^2} \|\nabla^2 (h^2)\|_{H^2} + \|\nabla^2 u\|_{L^3} \|\nabla^2 (h^2)\|_{L^6} + \|\nabla u\|_{L^\infty} \|\nabla^3 (h^2)\|_{L^2} \|\nabla^3 (h^2)\|_{L^2} \right) \\
+ C \left( 1 + \|\nabla^2 u\|_{L^2} + \|\nabla^3 (h^2)\|_{L^2} \right) \|\nabla^3 (h^2)\|_{L^2} \\
\leq C + C \|\nabla u_t\|_{H^1} + C \|\nabla^3 (h^2)\|_{L^2},
\]
which, together with Gronwall’s inequality and \(\text{sp26}\), yields that
\[
\sup_{0 \leq t \leq T} \|\nabla^3 (h^2)\|_{L^2} \leq C. \tag{2.60} \text{sp26}
\]
Collecting all these estimates \(\text{sp24},\text{sp26},\text{sp27}\), and \(\text{sp27}\) shows
\[
\sup_{0 \leq t \leq T} \|h^2 - \tilde{h}^2\|_{H^3} + \int_0^T \|\nabla u\|^2_{H^1} \, dt \leq C. \tag{2.61} \text{sp27}
\]
It is easy to check similar arguments work for \(h - \tilde{h}\) by using \(\text{sp27}\). Hence,
\[
\sup_{0 \leq t \leq T} \|h - \tilde{h}\|_{H^3} \leq C. \tag{2.62} \text{sp28}
\]
Combining (2.54), (2.56), (2.63), and (2.65), we derive
\[
\sup_{t \in [0,T]} (\|u_t\|_{H^2} + \|\nabla u\|_{H^2}) + \int_0^T (\|\nabla u\|_{H^3} + \|\nabla u_t\|_{H^1}) dt \leq C. \tag{2.63} \]

Furthermore, according to (2.63), (2.64), and (2.65), we have for any $0 < \tau < T$,
\[
\sup_{\tau \leq t \leq T} \|u_t\|_{H^2} \leq C(\tau). \tag{2.64} \]
Thus,
\[
\sup_{\tau \leq t \leq T} (\|u_t\|_{H^2} + \|\nabla^4 u\|_{L_2}) + \int_\tau^T \|\nabla u_t\|^2 dx dt \leq C(\tau), \tag{2.65} \]
due to (2.56), (2.64), and (2.65).

Finally, for any $0 < \tau < T$ with $T$ finite, due to the standard embedding
\[
L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^q), \quad \text{for any } q \in [2, \infty),
\]

it follows from (2.56), (2.64), (2.65), and (2.68) that
\[
\nabla u_t, \nabla^3 u \in C([\tau, T]; L^p), \quad \nabla u, \nabla^2 u \in C\left([\tau, T]; L^2 \cap C\left(\mathbb{R}^2\right)\right),
\]
for all $p \geq 2$. We are done.

**Step 4** It remains to prove the two claims (2.51) and (2.53).

First, we prove (2.51) by using Beal-Kato-Majda type inequality as in [22] (see [2] for the case $\text{div} u = 0$). For $p \geq 2$, $|\nabla h|^p$ satisfies
\[
(|\nabla h|^p)_t + \text{div}(|\nabla h|^p u) + (p-1)|\nabla h|^p \text{div} u
+ p|\nabla h|^{p-2}(\nabla h)^t \nabla u(\nabla h) + ph|\nabla h|^{p-2} \nabla h \cdot \nabla \text{div} u = 0.
\]
Thus,
\[
\partial_t |\nabla h|_{L^p} \leq C(1 + \|\nabla u\|_{L^\infty})|\nabla h|_{L^p} + C|\nabla^2 u|_{L^p}
\leq C(1 + \|\nabla u\|_{L^\infty})|\nabla h|_{L^p} + C|h^{1/2} u|_{L^2} + C|\nabla^2 u|_{L^2}, \tag{2.66}
\]
due to
\[
|\nabla^2 u|_{L^p} \leq C\left(\|h u\|_{L^p} + \|\nabla(h^2)|_{L^p} + \|hu^\perp\|_{L^p}\right)
\leq C|h^{1/2} u|_{L^2} + C|\nabla^2 u|_{L^2} + C|\nabla h|_{L^p} + C, \tag{2.67}
\]
which follows from the standard $L^p$-estimate for the following elliptic system:
\[
\mu \Delta u + (\mu + \lambda) \nabla \text{div} u = h \dot{u} + \nabla (h^2) + hu^\perp.
\]
The Beal-Kato-Majda type inequality and (2.57) give that for $q > 2$,
\[
|\nabla u|_{L^\infty}
\leq C (\|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + |\nabla^2 u|_{L^q}) + C|\nabla u|_{L^2} + C
\leq C (\|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + |h^{1/2} u|_{L^2} + |\nabla u|_{L^2} + \|\nabla h\|_{L^q}) + C
\leq C (\|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + |h^{1/2} u|_{L^2} + |\nabla u|_{L^2})
+ C (\|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + |\nabla h|_{L^q}) + C. \tag{2.68}
\]
Set
\[
\begin{aligned}
f(t) & \triangleq e + \|\nabla h\|_{L^q}, \\
g(t) & \triangleq (1 + \|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + \|h^{1/2} \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2}) \\
& \quad + \|h^{1/2} \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2}.
\end{aligned}
\]

Combining (2.68) with (2.66) and setting \(p = q\) in (2.66), one gets
\[
f'(t) \leq C g(t) f(t) + C g(t) f(t) \ln f(t) + C g(t),
\]
which yields
\[
(\ln f(t))' \leq C g(t) + C g(t) \ln f(t),
\]
due to \(f(t) > 1\). Note that (2.66), (2.68) and (2.70) imply
\[
\int_0^T g(t) dt \leq C \int_0^T \left( \|\text{div} u\|^2_{L^\infty} + \|\omega\|^2_{L^\infty} \right) dt + C
\]
\[
\leq C \int_0^T \left( \|F\|^2_{L^2} + \|\nabla F\|^2_{L^6} + \|\omega\|^2_{L^2} + \|\nabla \omega\|^2_{L^6} \right) dt + C
\]
\[
\leq C \int_0^T \left( \|h^{1/2} \dot{u}\|^2_{L^2} + \|\nabla \dot{u}\|^2_{L^2} + \|h^{1/2} u\|^2_{L^2} + \|\nabla u\|^2_{L^2} \right) dt + C
\]
\[
\leq C,
\]
which, together with (2.69) and Gronwall’s inequality, shows that
\[
\sup_{0 \leq t \leq T} f(t) \leq C.
\]
Consequently,
\[
(2.71) \quad \sup_{0 \leq t \leq T} \|\nabla h\|_{L^q} \leq C.
\]
As a consequence of (2.68), (2.70) and (2.71), one obtains
\[
\int_0^T \|\nabla u\|_{L^\infty} dt \leq C.
\]
(2.72)

Next, taking \(p = 2\) in (2.66), one gets by using (2.72), and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\nabla h\|_{L^2} \leq C,
\]
which, together with (2.67), (2.45), (2.71) and (2.72), gives (2.46).
Next, we derive (2.12). Differentiating (2.11) with respect to \(t\), then multiplying the resulting equation by \(u_t\), one gets after integration by parts that
\[
\frac{d}{dt} \int u^2 \nabla u_t^2 + \frac{\lambda + \mu}{2} (\text{div} u_t)^2 dx + \int h u_t^2 dx = \frac{d}{dt} \left( -\frac{1}{2} \int h_t |u_t|^2 dx - \int h_t u_t \cdot u_t dx + \int (h^2)_t \text{div} u_t dx \right) + \frac{1}{2} \int h_{tt} |u_t|^2 dx + \int (h u_t \cdot \nabla u_t) \cdot u_t dx - \int h u_t \cdot \nabla u_t u_t dx - \int (h^2)_t \text{div} u_t dx - \int (h u^2)_t u_t dx \tag{2.73}
\]
\[
= \frac{d}{dt} \left[ \frac{1}{2} I_0 + \sum_{i=1}^{\infty} I_i \right].
\]
Noticing that (2.20) and (2.21) yield that for any \(p \geq 2\)
\[
\|u_t\|_{L^p} \leq C \|u_t\|_{L^2} + \|\nabla u_t\|_{L^2} \leq C \|h^{1/2} u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}
\]
\[
\tag{2.74}
\]
\[\leq C + C \|\nabla u_t\|_{L^2}, \tag{2.75}
\]
it follows from (2.12), (2.14), (2.23), (2.24) and (2.25) that
\[
|I_0| = \left| \int \text{div}(hu) |u_t|^2 dx \right| + C \|h_t\|_{L^1} \|u \cdot \nabla u\|_{L^2} \|u_t\|_{L^6}
\]
\[
+ C \|(h^2)_t\|_{L^2} \|\nabla u_t\|_{L^2}
\]
\[
\leq C \int h |u| |u_t| |\nabla u_t| dx + C \|\nabla u_t\|_{L^2}
\]
\[
\leq C \|u\|_{L^p} \|h^{1/2} u_t\|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}
\]
\[
\leq \delta \|\nabla u_t\|_{L^2}^2 + C \delta,
\]
\[
2|I_1| = \left| \int (h_{tt} u + h u_t) \cdot \nabla (|u_t|^2) dx \right|
\]
\[
\leq C \left( \|h_t\|_{L^2} \|u\|_{L^\infty} + \|h^{1/2} u_t\|_{L^2} \|u_t\|_{L^6} \right) \|u_t\|_{L^6} \|\nabla u_t\|_{L^2}
\]
\[
\leq C \|\nabla u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^{5/2} + C
\]
\[
\leq C \|\nabla u_t\|_{L^2}^4 + C,
\]
and
\[
|I_2| = \left| \int (h_{tt} u \cdot \nabla u \cdot u_t + h_{tt} u_t \cdot \nabla u \cdot u_t + h_t u \cdot \nabla u_t \cdot u_t) dx \right|
\]
\[
\leq \|h_{tt}\|_{L^2} \|u \cdot \nabla u\|_{L^2} \|u_t\|_{L^6} + \|h_t\|_{L^2} \|u_t\|_{L^6} \|\nabla u\|_{L^6}
\]
\[
+ \|h_{tt}\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^6}
\]
\[
\leq C \|h_{tt}\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 + C.
\]
Cauchy’s inequality gives
integration by parts that multiplying \((2.79)\) and \((2.78)\)

Next, differentiate \((2.80)\), \((2.81)\) and \((2.82)\) that

Moreover, we observe from \((2.83)\) that

Next, differentiate \((2.80)\) with respect to \(t\) twist to get

Multiplying \((2.81)\) by \(u_t\) and then integrating the resulting equation over \(\mathbb{R}^2\), one gets after integration by parts that

We estimate each \(J_i(i = 1, \cdots, 6)\) as follows:

Hölder’s inequality gives

\[
|J_3| + |J_4| = \left| \int hu_t \cdot \nabla u \cdot u_t \, dx \right| + \left| \int hu_t \cdot \nabla u_t \, dx \right|
\leq C \|h^{1/2}u_t\|_{L^2} \left( \|u_t\|_{L^6} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \right)
\leq \delta \|h^{1/2}u_t\|_{L^2}^2 + C_\delta \|\nabla u_t\|_{L^2}^2 + C_\delta,
\]

\[
|J_5| \leq \|(h^2)u_t\|_{L^2} \|\nabla u_t\|_{L^2} \leq C \|(h^2)u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2.
\]
It follows from (2.41), (2.45), (2.49), and (2.50) that
\[
|J_2| \leq C\left(||h_{tt}||_{L^2} + ||h_t||_{L^2}\right)\left(||u_{tt}||_{L^2}||\nabla u_t||_{L^2} + ||u_{tt}||_{L^2}||u_t||_{L^6}\right)
\leq C\left(||h^{1/2}u_t||_{L^2}^{1/2}||u_t||_{L^6} + ||h_t||_{L^6}||u_t||_{L^6}\right)||\nabla u_{tt}||_{L^2}||\nabla u_t||_{L^2}
\leq C(1 + ||\nabla u_t||_{L^2}^{1/2})||\nabla u_{tt}||_{L^2}||\nabla u_t||_{L^2}
\leq \delta ||\nabla u_{tt}||_{L^2}^{2} + C_\delta (1 + ||\nabla u_t||_{L^2}^{3/2})||\nabla u_t||_{L^2}^{2} + C_\delta, \tag{2.84}
\]

\[
|J_3| \leq C\left(||h_{tt}||_{L^2}||u||_{L^\infty}||\nabla u||_{L^1} + ||h_t||_{L^6}||u||_{L^6}||\nabla u||_{L^2}\right) ||u_t||_{L^6}
\leq \delta ||\nabla u_{tt}||_{L^2}^{2} + C_\delta ||h_{tt}||_{L^2}^{2} + C_\delta ||\nabla u_t||_{L^2}^{2} + C_\delta, \tag{2.85}
\]

and
\[
|J_4| + |J_5| \leq C ||h_{tt}||_{L^2} ||\nabla u||_{L^1} ||u||_{L^6} + C \left(||h^2||_{tt}||u_{tt}||_{L^2}\right) ||u_{tt}||_{L^2}
\leq \delta ||\nabla u_{tt}||_{L^2}^{2} + C_\delta ||h^{1/2}u_t||_{L^2}^{2} + C_\delta (||h^2||_{tt}||u_t||_{L^2}^{2}). \tag{2.86}
\]

Moreover, we observe that
\[
|J_6| \leq C \left(||h_{tt}||_{L^2}||u||_{L^\infty} + ||h_t||_{L^4}||u_t||_{L^2} + ||u||_{L^4}||u_t||_{L^2}\right) ||u_t||_{L^2}
\leq \delta ||\nabla u_{tt}||_{L^2}^{2} + C_\delta ||h_{tt}||_{L^2}^{2} + C_\delta ||\nabla u_t||_{L^2}^{2}. \tag{2.87}
\]

Combining (2.53)–(2.56) and (2.59)–(2.61), choosing \(\delta\) small enough, we have
\[
\frac{d}{dt} \int(|\nabla u_t|^2 + h|u_t|^2) \, dx + \int(hu_{tt}^2 + |\nabla u_{tt}|^2) \, dx
\leq C(1 + ||\nabla u_t||_{L^2}^{2})||\nabla u_t||_{L^2}^{2} + C||h_{tt}||_{L^2}^{2}
+ C||h^{1/2}u_t||_{L^2}^{2} + C(||h^2||_{tt}||u_t||_{L^2}^{2} + C. \tag{2.88}
\]

Due to the regularity of the local solution, \(u_t \in C([0,T_0]; L^2)\) and \(t^{1/2}\sqrt{u_{tt}} \in L^\infty([0,T_0]; L^2)\), we obtain
\[
||\nabla u_t(\cdot,T_0/2)||_{L^2} + \int h|u_{tt}|^2 \, dx(T_0/2)
\leq \frac{2}{T_0} \left(||t\nabla u_{tt}||_{L^\infty(0,T_0/2)} + ||t^{1/2}\sqrt{h_{tt}}||_{L^\infty(0,T_0/2)}\right) \tag{2.89}
\leq C,
\]

where \(C\) may also depend on ||\nabla g||_{L^2}. Therefore, we deduce from (2.88), (2.89), and Gronwall's inequality that
\[
\sup_{T_0/2 \leq t \leq T} \left(||\nabla u_t||_{L^2} \int h|u_{tt}|^2 \, dx + \int_{T_0/2}^{T} (h|u_{tt}|^2 + |\nabla u_{tt}|^2) \, dx \right) \leq C. \tag{2.90}
\]

On the other hand, (2.47) gives
\[
\sup_{0 \leq t \leq T_0/2} ||\nabla u_{tt}||_{L^2} + \int_{0}^{T_0/2} h|u_{tt}|^2 \, dx \leq C. \tag{2.91}
\]

Combining (2.90), (2.91) yields (2.41) immediately.

Furthermore, for any \(\tau \in (0,T_0)\), there exists some \(t_0 \in (\tau/2, \tau)\) such that
\[
||\nabla u_t(\cdot,t_0)||_{L^2} + \int h|u_{tt}|^2 \, dx(t_0) \leq C(\tau).
\]
Similarly as in Corollary 2.5, Gronwall’s inequality gives
\[
\sup_{0 \leq t \leq T} \int h|u_t|^2 dx + \int_0^T \int |\nabla u_t|^2 dx dt \leq C(\tau).
\]
Hence Lemma 3.1 holds due to \( t_0 < \tau \).

3. PROOF OF THEOREM 1.2

With Lemma 2.3 and Lemma 2.0 at hand, we are able to prove theorem 1.2 in this section.

**Proof.** By theorem 1.1, there exists a \( T_0 > 0 \) such that the Cauchy problem (1.9), (1.5) and (1.1) has a unique classical solution \((h, u)\) on \((0, T_0]\). By Lemma 2.0 and Lemma 4.0, the local classical solution can be extended to all time. Let

\[
T^* = \sup \{ T \mid (h, u) \text{ is a classical solution on } (0, T) \} \tag{3.1}
\]

We argue by contradiction. We may assume \( T^* < \infty \).

Since
\[
A_1(0) + A_2(0) = 0, \quad h_0 \leq \bar{h},
\]
there exists a \( t_1 \in (0, T^*) \) such that (2.24) holds for \( T = t_1 \). For any \( 0 < \tau < t_1 < T^* \), according to (2.47), (2.48), and (2.24), we have
\[
\int_\tau^{t_1} \| \langle h \rangle |u_t|^2 \rangle_L \, dt
\]
\[
\leq \int_\tau^{t_1} \left( \| \dot{h} \| |u_t|^2 \rangle_L + 2 \| hu_t \cdot ut \rangle_L \right) \, dt
\]
\[
\leq C \int_\tau^{t_1} \left( \| |\text{div} h| |u_t|^2 \rangle_L + \| u \| |\nabla h| |u_t|^2 \rangle_L + \| h^{1/2} u_t \| \langle L2 \| h^{1/2} u_t \rangle_L \right) \, dt
\]
\[
\leq C \int_\tau^{t_1} \left( \| |u_t|^2 \rangle_L \| \nabla u \| \langle L\infty + \| u \| \langle L6 \| \nabla h \| \langle L2 \| u_t \| \langle L6 + \| h^{1/2} u_t \| \langle L2 \right) \, dt
\]
\[
\leq C.
\]

Thus
\[
h^{1/2} u_t \in C(\tau, t_1; L^2),
\]
which together with Lemma 2.4 implies that
\[
h^{1/2} \dot{u}, \nabla \dot{u} \in C(\tau, t_1; L^2).
\]

In fact \((h(x, t_1), u(x, t_1))\) satisfies (1.9) and (1.5) with \( g(x) = \dot{u}(x, t_1) \). Therefore, by theorem 1.1 and Lemma 2.0, there exist a \( t_2 > 0 \) such that we are able to extend \( t_1 \) to \( t_1 + t_2 \) with (2.24) still holding for \( T = t_1 + t_2 \), \( t_0 \) only depends on \( u, \lambda, \bar{h}, \tilde{h} \) and \( K \).

By finite times, there exist \( T_1 > T^* \) such that \((h, u)\) is a classical solution on \((0, T_1]\), which is a contradiction to (3.1). Then, we are done.

**Remark 3.1.** We only prove the global existence in case \( \tilde{h} > 0 \). When the initial height has compact support, there is a blow up example in [21].

**Acknowledgments** Ben would like to thank Professor Yuxi Zheng for his interests in this work and for stimulating discussions.
 References


