§1. Strong approximation and integral pts.

(1.1) strong approximation.

- let k be a number field Ω_k be the set of all primes of k.
- ∞_k the set of all arch primes · $\mathcal{O}_k$ ring of integers.
- X separated sch of finite type over k.

(1.1.1) Def A separated sch $\mathfrak{X}$ of finite type / $\mathcal{O}_k$ is called integral model of X if $\mathfrak{X} \times_{\mathcal{O}_k} k = X$

- Fix an integral model $\mathfrak{X}$ of X, one can define

$$X(\mathbb{A}_k) \cong \bigcup_{\substack{S \text{ finite} \\ \cup \\ \infty_k}} \prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathfrak{X}(\mathcal{O}_{k_v}) \leftarrow \text{indep of choice of } \mathfrak{X}.$$

$$X(\mathbb{A}_k^f) \cong \bigcup_{S \text{ finite}} \prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathfrak{X}(\mathcal{O}_{k_v})$$

(1.1.2) Def X satisfies strong approximation if $X(k)$ is dense in $X(\mathbb{A}_k^f)$ ← say 'SA'

Example. $X = \mathbb{G}_a^n$, SL_n, $Sp(n)$. Spin with $Spin(k_\infty)$ non-compact

(1.1.3) Thm

Let X semi-simple, simply conn LAG. Then, X satisfies SA iff $G'(k_\infty)$ is not compact for any simple factor G'/k of X.

(1.1.4) Diophantine interpretation.

Prop X satisfies SA $\Leftrightarrow$ for any integral model $\mathfrak{X}$ of X

$$\mathfrak{X}(\mathcal{O}_k) \neq \phi \Leftrightarrow \prod_{v \in \Omega_k} \mathfrak{X}(\mathcal{O}_{k_v}) \neq \phi. \qquad \text{(Hasse principle)}$$

Sketch of proof) for any open set $W \subseteq X(k_v)$. $\exists$ integral model $\mathcal{Z}$ of X s.t $\mathcal{Z}(\mathcal{O}_{k_v}) \subseteq W$. + Gluing process.

(1.1.5) Thm (M cor) Supp X is smth and geometrically integral.

If X satisfies SA, then $X_{\bar{k}}$ is simply connected, $\pi_1^{et}(X_{\bar{k}}) = \{1\}$. ← no finite étale covering.

(Note: Fund gp of alg gp, var could be different, but simply conn is same)

Example. $X = \mathbb{G}_m$. Since $\mathbb{G}_m$ is not simply conn $/\bar{k}$, Then $\mathbb{G}_m$ does not satisfies sA. Namely, $k^\times$ is not dense in $\mathbb{I}_k^f$.

(1.2) Strong approximation with Brauer-Manin obstruction.

(1.2.2) Def X satisfies SA with Brauer-Manin obstruction if $X(k)$ is dense in $\mathrm{Pr}_f(X(\mathbb{A}_k)^{Br})$ where $\mathrm{pr}_f : X(\mathbb{A}_k) \to X(\mathbb{A}_k^f)$ the projection map.

(1.2.3) Example. $X = \mathbb{G}_m$. Then, X satisfies SABM.

pf) By the spectral sequence,

$$H^p(k, H^q(X_{\bar{k}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$$

$$\Rightarrow \quad 0 \to H^1(k, \bar{k}(X)^\times) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$$

$$\to H^2(k, \bar{k}(X)^\times) \to \mathrm{Br}_1(X) \to H^1(k, \mathrm{Pic}(X_{\bar{k}}))$$

Since $\underset{\mathrm{Spec}\,k[t,t^{-1}]}{X \overset{\shortparallel}{=} \mathbb{G}_m}$, $\bar{k}(X)^\times = \bar{k}^\times \times \langle t \rangle^{\mathbb{Z}}$. $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\bar{k}}) = 0$.

One obtain, $0 \to H^2(k, \bar{k}^\times \cdot \langle t \rangle^{\mathbb{Z}}) \to \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{\bar{k}}))$ real contribution.

By Tsen-Lang, $\mathrm{Br}(X_{\bar{k}}) = 0$, one concludes $\underset{\mathrm{Br}(k).}{\underset{\shortparallel}{H^2(k, \bar{k}^\times)}} \oplus H^2(k, \mathbb{Z}) \cong \mathrm{Br}(X)$

$$H^2(k, \mathbb{Z}) \cong H^1(G_k, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}).$$

Then, $\forall \xi \in H^2(k, \mathbb{Z})$, $x_v \in X(k_v) = k_v^\times$ $\mathrm{inv}_v(\xi(x_v)) = \xi(\rho_v(x_v)) \in \mathbb{Q}/\mathbb{Z}$

where $\rho_v : k_v^\times \to \mathrm{Gal}(k_v^{ab}/k_v) \hookrightarrow \mathrm{Gal}(k^{ab}/k)$ local Artin map.

This implies $X(\mathbb{A}_k)^{Br(X)} = \{(x_v) \in \mathbb{I}_k,\ \underset{\text{in } \mathbb{Q}/\mathbb{Z}}{y(\rho((x_v))) = 0} \ \forall y \in \mathrm{Hom}_{cts}(\mathrm{Gal}(k^{ab}/k), \mathbb{Q}/\mathbb{Z})$

where $\rho : \mathbb{I}_k \to \mathrm{Gal}(k^{ab}/k)$ the global Artin map. $(\rho = \pi' \rho_v)$

(This is because, $\mathrm{Br}(k)$ plays no role. $\because$ $0 \to \mathrm{Br}(k) \to \oplus \mathrm{Br}(k_v) \xrightarrow{\Sigma \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$ exact)

By Pontryagin duality,

$$X(\mathbb{A}_k)^{Br(X)} = \{(x_v) \in \mathbb{I}_k,\ \rho((x_v)) = 1 \in \mathrm{Gal}(k^{ab}/k)\} = \ker(\rho)$$

$\ker(\rho) = \overline{k^\times \cdot k_\infty^f}$ $\Rightarrow$ $k^\times$ is dense in $\mathrm{Pr}_f(\ker(\rho))$

By CFT.

(1.2.4) Diophantine Interpretation.

Prop X satisfies SABM $\Leftrightarrow$ for any integral model $\mathcal{X}$ of X,

$$\mathcal{X}(\mathcal{O}_k) \neq \phi \iff \left(\prod_{v \in \Omega_k} \mathcal{X}(\mathcal{O}_{k_v})\right)^{Br(X)} \neq \phi.$$

(1.2.5) Let G be a conn LAG $/k$.

- $R(G)$ = max and $\overset{}{\underset{\text{conn}}{}}$solv normal subgp of G/k.
- $G^{ss} = G/R(G)$ a semi-simple LAG$/k$.

Thm (B — D , 2013)

Let $X = G/H$, where G and H are conn LAG $/k$.

If $G'(k_\infty)$ is not compact for any simple factor G' of G^{ss}, then X satisfies SABM.

(1.2.6) Example. Let $A, B \in M(\mathcal{O}_k)$. Determine exactly when A and B are similar over $\mathcal{O}_k$

i.e. $\exists T \in GL_n(\mathcal{O}_k)$ s.t $A = T^{-1}BT$.

(In general, this problem does not satisfy Hasse-principle)

Even the char poly $p(\lambda) = \det(\lambda I_n - A)$ is irred

M. Newman, Integral matrices ($\mathbb{Z}$)

Step I. Determine A is similar to B over k (known)

$\Leftrightarrow$ $\lambda I_n - A$ is equivalent to $\lambda I_n - B$ over $k[\lambda]$.

Step II. Assume A is similar to B $/k$. Let $C_{GL_n}(A) = \{g \in GL_n : gA = Ag\}$

is a conn LAG. Then, $XA = BX$, $\det(X) \neq 0$. is a trivial torsor $/k$

under $C_{GL_n}(A)$. $\mathcal{X}A = B\mathcal{X}$, $\det(\mathcal{X}) \in \mathcal{O}_k^\times$ is an integral model of this torsor

By the above theorem, the above eqn has integral solution determined by BM obstruction.

§ Tamagawa measures

(2.1) Invariant differential forms and volume forms

(2.1.1) Let G be LAG / k.

- $\Delta^*: k[G]\otimes k[G] \to k[G]$. let $I = \ker \Delta^*$. Then, $\Omega^1_{G/k} := I/I^2$ locally free $k[G]$ mod
- $\forall g \in G(k)$, $T_g: G \to G$, $x \mapsto gx$. $g^*: \Omega^1_{G/k} \cong \Omega^1_{G/k}$ as $k[G]$-module.
- Define $\mathrm{Lie}(G) = \{\omega \in \Omega^1_G : g^*\omega = \omega,\ \forall g \in G(k)\}$. Is a vec sp of $\dim(G)$ over k.

(2.1.2) Examples

- $G = \mathbb{G}_m = \mathrm{Spec}\, k[t, \frac{1}{t}]$. $\Omega^1_{G/k} = k[t,\frac{1}{t}]dt$, $\mathrm{Lie}(G) = k\left(\frac{dt}{t}\right)$ Inv form. $\begin{cases} t \mapsto g^*t = ct \\ \frac{dt}{t} \mapsto \frac{dg^*t}{g^*t} = \frac{dt}{t}. \end{cases}$
- $G = GL_n = \mathrm{Spec}\, k[x_{ij}, \frac{1}{\det(x_{ij})}]_{1\le i,j\le n}$. Then, $\Omega^1_{G/k} = \bigoplus_{1\le i,j\le n} k[x_{ij}, \frac{1}{\det(x_{ij})}]\, dx_{ij}$

what is $\mathrm{Lie}(G) = ?$. let $X = (x_{ij})$. Since $X^{-1}dX$ is invariant under matrix multiplication,

one has $\mathrm{Lie}(G) = \bigoplus_{1\le i,j\le n} k\left(\frac{1}{\det(x_{ij})}\sum_{k=1}^{n} A_{ki}\, dx_{kj}\right)$ A_{ki} = cofactor.

For $n=2$, $X^{-1}dX = \frac{1}{\det(x_{ij})}\begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11}\end{pmatrix}\begin{pmatrix} dx_{11} & dx_{12} \\ dx_{21} & dx_{22}\end{pmatrix}$

- $G = SL_2$, $\mathrm{Lie}(G) = k\omega_1 \oplus k\omega_2 \oplus k\omega_3 = \mathfrak{sl}_2$

(2.1.3) Def $\bigwedge_{i=1}^{\dim(G)} \mathrm{Lie}(G)$ is one dim'l vec sp. Each nonzero elt in this space is called 'volume form'.

Example. 1. $G = GL_2$. ; $\frac{1}{\det(x_{ij})^2}\, dx_{11}\wedge dx_{12}\wedge dx_{21}\wedge dx_{22}$

2. $G = GL_n$: $\frac{1}{\det(x_{ij})^n} \bigwedge_{1\le i,j\le n} dx_{ij}$

3. $G = SL_2$: $= \begin{cases} \frac{1}{x_{11}}(dx_{11}\wedge dx_{21}\wedge dx_{12}) & x_{11}\neq 0 \\ \frac{1}{x_{21}}(dx_{11}\wedge dx_{21}\wedge dx_{22}) & x_{21}\neq 0. \end{cases}$

(2.2) Tamagawa measure.

(2.2.1) Thm (Haar) For any loc cpt top'l gp, there is a unique left invariant measure up to a scalar. (called Haar measure)

(2.2.2) let G be a LAG / k and ω be a volume form of G.

For any $v \in \Omega_k$, $G(k_v)$ is loc cpt top'l gp.

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Let $\mathcal{U}$ be an open set of G s.t $\omega|_{\mathcal{U}} = f(x_1,\cdots,x_n)\, dx_1\wedge\cdots\wedge dx_n$

If $v \notin \infty_k$, for any open compact subset C of $\mathcal{U}(k_v)$, one can define a measure μ by

$$\mu(C) = \int_C |f(x_1,\cdots,x_n)|\, dx_1\wedge\cdots\wedge dx_n$$

with normalization $\int_{\mathcal{O}_{k_v}} dx = [\mathcal{O}_{k_v} : \mathcal{D}_{k_v/\mathbb{Q}_p}]^{-\frac{1}{2}}$ where $\mathcal{D}_{k_v/\mathbb{Q}_p}$ is different ideal for $k_v/\mathbb{Q}_p$.

For $v \in \infty_k$, one can take the Lebesgue measure.

Since ω is left invariant, the measure defined above is the Haar measure.

(2.2.3) Example: $G = \mathbb{G}_m$:

$$\int_{\mathbb{G}_m(\mathbb{Z}_p)} \frac{dt}{|t|_p} = \int_{\mathbb{Z}_p^\times} dt = \sum_{i=1}^{p-1} \int_{i+p\mathbb{Z}_p} dt = (p-1)\cdot\frac{1}{p} = (1-\frac{1}{p}).$$

- $G = SL_2$: See (2.1.3)-3

$$\int_{SL_2(\mathbb{Z}_p)} \omega = \#(SL_2(\mathbb{F}_p)) \int_H \omega = \# SL_2(\mathbb{F}_p) \int_{1+p\mathbb{Z}_p} \underbrace{\frac{dx_{11}}{|x_{11}|_p}}_{1} \int_{p\mathbb{Z}_p} dx_{12} \int_{p\mathbb{Z}_p} dx_{21}$$

$$= (1-\frac{1}{p^2})$$

Since $1 \to H \to SL_2(\mathbb{Z}_p) \to SL_2(\mathbb{F}_p) \to 1$. $\qquad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix} = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$ x_1, etc

$\quad\quad\; \{I + M_2(p\mathbb{Z}_p)\}$

(2.2.4) Let G be LAG. Define $\hat{G}(k) := \mathrm{Hom}_{k\text{-alg}}(G, \mathbb{G}_m)$

$\cong k[G]^\times / k^\times$ (Rosenlicht lemma)

$=$ a free abelian group of finite rank.

If G is semi-simple, then $\hat{G}(k) = \{1\}$

Thm If G is LAG$/k$ with $\hat{G}(k)$, then $\prod_{v<\infty_k} \int_{K_v} \omega$ is convergent when $\prod_{v<\infty} K_v$ is an open compact subgp of $G(\mathbb{A}_k^f)$.

(almost all v, $K_v = G(\mathcal{O}_{k_v})$. $G/\mathcal{O}_{k,S}$ integral model of G.)

- $G = \mathbb{G}_m \rightsquigarrow \exists$ nontrivial char + $\prod \int \omega = \prod(1-\frac{1}{p})$ not conv
- $G = SL_2 \rightsquigarrow$ semisimple + $\prod \int \omega = \prod(1-\frac{1}{p^2})$ converges.

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- The measure over $G(\mathbb{A}_k)$ defined by the product of the local measures.
- By the product formula, the Tamagawa measure on $G(\mathbb{A}_k)$ is independent of the choice of ω.

(2.2.5) Let G be a conn LAG$/k$. Then, $G(k)$ is discrete in $G(\mathbb{A}_k)$

Thm (Weil-Langlands-S — Kottwitz)

If $\hat{G}(k) = \{1\}$, then $T(G) = \int_{G(k)\backslash G(\mathbb{A}_k)} \omega = \dfrac{\# \mathrm{Pic}(G)}{\# Ш^1(k, G)}$

where $Ш^1(k, G) = \ker\left(H^1(k, G) \longrightarrow \prod_{v \in \Omega_k} H^1(k_v, G) \right)$

Example : $G = SL_2 / \mathbb{Q}$ $\quad T(G) = 1$ $\quad (\because \mathrm{Pic}(G) = Ш^1(\overset{k}{\mathbb{Q}}, G) = 1 \}$.

← for field is UFD.

Then, $\int_{SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}_\mathbb{Q})} = 1$

By strong approximation, $SL_2(\mathbb{A}_\mathbb{Q}) = SL_2(\mathbb{Q}) \cdot (SL_2(\mathbb{R}) \times \prod_p SL_2(\mathbb{Z}_p))$

$$SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}_\mathbb{Q}) = \frac{SL_2(\mathbb{R})}{SL_2(\mathbb{Z})} \times \prod_p SL_2(\mathbb{Z}_p)$$

$$\mathrm{vol}\left(\frac{SL_2(\mathbb{R})}{SL_2(\mathbb{Z})}\right) \cdot \prod_p (1 - p^{-2}) = 1 \Rightarrow \mathrm{Vol}\left(\frac{SL_2(\mathbb{R})}{SL_2(\mathbb{Z})}\right) = \zeta(2)$$

Example : Let f be a non-degenerated quadratic forms over $\mathbb{Z}$.

and $G = SO(f)$ semisimple over $\mathbb{Q}$.

$$\mathrm{gen}(f) = \{ g : g \sim_{\mathbb{Z}_p} f \ v \in \Omega_\mathbb{Q}, \mathbb{Z}_\infty = \mathbb{R} \}$$

$$\updownarrow 1\text{-}1$$

$$SO(f) \backslash SO_\mathbb{A}(f) / \prod_{v \in \Omega_k \in \mathbb{Q}} SO(f)(\mathbb{Z}_p)$$

Since $Ш^1(SO(f)) = 1$ (Hasse-Minkowski), $\mathrm{Pic}(SO(f)) = 2$,

$$T(SO(f)) = 2 = \int_{SO(f) \backslash SO_{\mathbb{A}_\mathbb{Q}}(f)} \omega = \sum_{i=1}^{g} \int_{SO(f) \backslash SO(f) \alpha_i \cdot \prod_{v \in \Omega_\mathbb{Q}} SO(f)(\mathbb{Z}_p)} \omega$$

$$= \sum_{i=1}^{g} \int_{SO(f_i) \backslash SO(f_i) \prod_{p \in \Omega_{\mathbb{Q}}} SO(f_i)(\mathbb{Z}_p)} \omega \qquad (f_i = g_i(f))$$

Since $SO(f_i)(k) \backslash SO(f_i)(k) \cdot \prod_{p \in \Omega_{\mathbb{Q}}} SO(f_i)(\mathbb{Z}_p) \cong \dfrac{SO(f_i)(\mathbb{R})}{SO(f_i)(\mathbb{Z})} \times \prod_p SO(f_i)(\mathbb{Z}_p)$.

$$2 = \sum_{i=1}^{g} \mathrm{vol}\left(\frac{SO(f_i)(\mathbb{R})}{SO(f_i)(\mathbb{Z})}\right) \prod_p \int_{SO(f_i)(\mathbb{Z}_p)} \omega$$

$$\mathrm{vol}\left(\frac{SO(f_i)(\mathbb{R})}{SO(f_i)(\mathbb{Z})}\right) = \frac{1}{\# SO(f_i)(\mathbb{Z})} \int_{SO(f_i)(\mathbb{R})} \omega \qquad (f \text{ is positive definite})$$

$$\therefore \sum_{i=1}^{g} \frac{1}{\# SO(f_i)(\mathbb{Z})} = 2 \cdot \prod_{p<\infty} \alpha_p^{-1}.$$

(2.3) Homogeneous space.

(2.3.1) Let G be LAG and H be a closed subgp of G over k

Thm (Chevalley) The fppf sheaf associated to the presheaf:

$$((k\text{-schemes})) \longrightarrow ((\text{Sets}))$$
$$A \longmapsto G(A)/H(A)$$

is representable. i.e. $\exists$ a scheme X over k such that $\mathrm{Mor}(-, X)$ is a sheaf associated to this presheaf.

Def We call this X is a homogeneous space of G, written as G/H.
Then, there is a quotient map $G \to G/H$ over k which is smooth.

(2.3.2) The quotient map $G \to G/H$ is a torsor under H. Then, one obtains the long exact sequence

$$1 \to H(k_v) \to G(k_v) \to (G/H)(k_v) \to H^1(k_v, H) \to H^1(k_v, G)$$

Thm $H^1(k_v, H)$ is finite for any $v \in \Omega_k$. (Platonov–Rapinchuk. Algebraic gps and number theory)

Hence, $(G/H)(k_v)$ has only finitely many $G(k_v)$-orbits.

(2.3.3) Thm Let G be a connected reductive group over k and H be a closed connected reductive subgroup of G. Then,

(i) (Matsushima) G/H is affine.

(ii) If $\hat{G}(k) = \hat{H}(k) = \{1\}$, then the Tamagawa measure ω_G and ω_H are unimodular (both left and right invariant)

Then, one can define the measure on $G(k_v)/H(k_v)$ as the quotient of top'l groups with the following formula.

$$\int_{G(k_v)} f(x)\,\omega_G(x) = \int_{G(k_v)/H(k_v)} \omega_{G/H}(g) \int_{H(k_v)} f(h\cdot g)\,\omega_H(h)$$

Since $(G/H)(k_v) = \bigcup_i G(k_v)x_i$, one has

$$1 \to H_i(k_v) \to G(k_v) \twoheadrightarrow \underset{\substack{\parallel \\ G(k_v)/H_i(k_v)}}{G(k_v)x_i} \subseteq (G/H)(k_v)$$ where H_i is the stabilizer of x_i.

§3. Counting integral points via SABM and equi-distribution.

(2.1) Set up.

- Let G be a connected reductive linear algebraic group and H be a closed connected reductive subgroup $/k$.
- Then, $X = G/H \overset{\iota}{\hookrightarrow} \operatorname{Spec} k[x_1, \cdots, x_n]$. $\forall x \in X(k) \subset k^n$, $x = (\alpha_1, \cdots, \alpha_n) \in k^n$.

<u>Def</u> $\|x\|_{\infty_k} = \sqrt{\sum_{i=1}^{n} \sum_{v \in \infty_k} \left(N_{k_v/\mathbb{R}}(\alpha_i)\right)^{e_i}}$, $e_i = \begin{cases} 1 & k_v = \mathbb{C} \\ 2 & k_v = \mathbb{R} \end{cases}$.

$\left(\text{or } \max_{v \in \infty_k} \max_{1 \le i \le n} \{|\alpha_i|_v\}\right)$

- Let $\mathcal{X}$ be an integral model of X over $\mathcal{O}_k$. Then, the set $\{x \in \mathcal{X}(\mathcal{O}_k) : \|x\|_\infty \le T\}$ is finite for any $T > 0$.

proof : Let $X(k_\infty, T) = \{x \in X(k_\infty) : \|x\|_\infty \le T\}$ is compact

Since $\{x \in \mathcal{X}(\mathcal{O}_k) : \|x\|_\infty \le T\} = X(k) \cap \underbrace{\{X(k_\infty, T) \times \prod_{v \notin \infty_k} \mathcal{X}(\mathcal{O}_{k_v})\}}_{\text{compact}}$

Therefore,

$N(\mathcal{X}, T) := \#\{x \in \mathcal{X}(\mathcal{O}_k), \|x\|_\infty \le T\} < +\infty$.

(Basic question.) $N(\mathcal{X}, T) \sim ?$ as $T \to \infty$.

(3.2) Arithmetic subgroups

(3.2.1) Local arithmetic groups.

- Fix a finite subset S of Ω_k containing ∞_k such that
  i) $\mathcal{H}$ and $\mathcal{G}$ are smooth group schemes of finite type over $\mathcal{O}_{k,S}$ and $\mathcal{H} \hookrightarrow \mathcal{G}$ over $\mathcal{O}_{k,S}$ with $\mathcal{H} \times_{\mathcal{O}_{k,S}} k = H \hookrightarrow G = \mathcal{G} \times_{\mathcal{O}_{k,S}} k$
  ii) $\mathcal{X} \times_{\mathcal{O}_k} \mathcal{O}_{k,S} \cong \mathcal{G}/\mathcal{H}$

<u>Def</u> $St_v(\mathcal{X}) = \begin{cases} \mathcal{G}(\mathcal{O}_{k_v}) & v \notin S \\ G(k_v) & v \in \infty_k \\ \{g \in \mathcal{G}_v(\mathcal{O}_{k_v}) : g \cdot \mathcal{X}(\mathcal{O}_{k_v}) = \mathcal{X}(\mathcal{O}_{k_v})\} & v \in S \setminus \infty_k \end{cases}$

where $\mathcal{G}_v$ is fixed smooth group scheme of finite type over $\mathcal{O}_{k,v}$ s.t. $\mathcal{G}_v \times_{\mathcal{O}_{k_v}} k_v = G \times_k k_v$

Then, $St_v(\mathfrak{X})$ acts on $\mathfrak{X}(\mathcal{O}_{k_v})$ for all $v \in \Omega_k$.

FACT The number of orbits $[St_v(\mathfrak{X}) \backslash \mathfrak{X}(\mathcal{O}_{k,v})] < +\infty$.

proof) $v \in \infty_k$: $\leq \# H^1(k_v, H)$

$v \notin \infty_k$: $\mathfrak{X}(\mathcal{O}_{k_v})$ is compact and each orbit is open

(3.2.2) Let $\Gamma = G(k) \cap \prod_{v \in \Omega_k} St_v(\mathfrak{X})$. Then, Γ acts on $\mathfrak{X}(\mathcal{O}_k)$.

Thm (Borel and Harish-chandra) The number of orbits $[\Gamma \backslash \mathfrak{X}(\mathcal{O}_k)]$ is finite.

Basic idea : $\mathfrak{X}(\mathcal{O}_k) = \bigcup_{i=1}^{h} \Gamma x_i$. Determine $h = ?$.

(3.3) Partition $\mathfrak{X}(\mathcal{O}_k)$ according to $St(\mathfrak{X}) = \prod_{v \in \Omega_k} St_v(\mathfrak{X})$

Def $\forall x, y \in \mathfrak{X}(\mathcal{O}_k)$. $x \sim y \iff \exists S_A \in St(\mathfrak{X})$ s.t $x = S_A \cdot y$.

The set of equivalence classes is denoted by $\mathfrak{X}(\mathcal{O}_k)/\sim$

Prop If G is semi-simple and simply connected such that $G'(k_\infty)$ is not compact for any simple factor G' of G, then the map induced by the diagonal map

$$\mathfrak{X}(\mathcal{O}_k)/\sim \xrightarrow{1-1} \Big(\prod_{v \in \Omega_k} \mathfrak{X}(\mathcal{O}_{k_v})\Big)^{Br(X)}_{St(\mathfrak{X})}$$ is bijective.

proof) Step 1. For any $(x_v) \in \prod_{v \in \Omega_k} \mathfrak{X}(\mathcal{O}_{k_v})$ and $S_A = (S_v) \in St(\mathfrak{X})$, one has

$$e_\xi((s_v)\cdot(x_v)) = e_{\varphi^*(\xi)}((s_v)) + e_\xi((x_v)) = e_\xi((x_v))$$

$$\left(G \xrightarrow{\varphi} X \iff \varphi^* : Br(X) \longrightarrow Br(G) \underset{\text{semi simple, simply conn}}{\overset{}{\cong}} Br(k) \right)$$

$$\xi \longmapsto \varphi^*(\xi)$$

$\Rightarrow St(\mathfrak{X})$ acts on $\Big(\prod_{v \in \Omega_k} \mathfrak{X}(\mathcal{O}_{k_v})\Big)^{Br(X)}$

• G semi-simple, simply connected / k. + $G'(k_\infty)$ is not compact for any simple factor G' of G.

• H connected reductive closed subgroup with $\hat{H}(k) = \{1\}$.

• $X = G/H$ $\mathfrak{X}$ integral model.

$N(\mathfrak{X}, T) := \#\{x \in \mathfrak{X}(O_k) : \|x\|_\infty \le T\}$. Asymptotic formula for $N(\mathfrak{X}, T)$, $T \to \infty$.

• $\Gamma = G(k) \cap St(\mathfrak{X}) \curvearrowright \mathfrak{X}(O_k)$, $St(\mathfrak{X}) = \prod_{v \in \Omega_k} St_v(\mathfrak{X})$. $\underline{\mathfrak{X}(O_k) = \dot{\bigcup}_{i=1}^{h} \Gamma x_i}$

Def For any $x, y \in \mathfrak{X}(O_k)$, $x \sim y \iff x = s_A \cdot y$, $s_A \in St(\mathfrak{X})$.

The set of equivalence classes is denoted by $\mathfrak{X}(O_k)/\sim$

Prop $\mathfrak{X}(O_k)/\sim \xrightarrow{\cong} \left(\prod_{v \in \Omega_k} \mathfrak{X}(O_{k_v}) \right)^{Br(X)} \Big/ St(\mathfrak{X})$

proof) Injectivity is clear and surjectivity follows from SABM (Borovoi - D

• $\mathfrak{X}(O_k) = \dot{\bigcup}_i (\mathfrak{X}(O_k) \cap St(\mathfrak{X}) \cdot x_i)$ $x_i \in \mathfrak{X}(O_k)$.

(3.4) Partition $\mathfrak{X}(O_k) \cap St(\mathfrak{X}) x_i$ according to $G(k)$.

For any $y, \xi \in \mathfrak{X}(O_k) \cap St(\mathfrak{X}) x_i$, we say $y \sim_G \xi$ if $\exists \sigma \in G(k)$ s.t $y = \sigma \cdot \xi$.

The set of equivalence classes is denoted by $(\mathfrak{X}(O_k) \cap St(\mathfrak{X}) x_i)/\sim_G$

Prop $\# \left(\dfrac{\mathfrak{X}(O_k) \cap St(\mathfrak{X}) x_i}{\sim_G} \right) = \# Ш(k, H_i)$ where H_i is the stabilizer of x_i in G.

proof) By the exact sequence.

$$\begin{tikzcd}
1 \arrow[r] & H_i(k) \arrow[r] \arrow[d] & G(k) \arrow[r] \arrow[d] & X(k) \arrow[r, "\partial x_i"] \arrow[d] & H^1(k, H_i) \arrow[r] \arrow[d] & H^1(k, G) \arrow[d] \\
1 \arrow[r] & H_i(A_k) \arrow[r] & G(A_k) \arrow[r] & X(A_k) \arrow[r, "\prod \partial^v_{x_i}"] & \bigoplus_{v \in \Omega_k} H^1(k_v, H_i) \arrow[r] & \bigoplus_{v \in \Omega_k} H^1(k_v, G)
\end{tikzcd}$$

Then, $\mathfrak{X}(O_k) \cap St(\mathfrak{X}) x_i /\sim_G \xhookrightarrow{\partial x_i} \ker \left(H^1(k, H_i) \longrightarrow \bigoplus_{v \in \Omega_k} H^1(k_v, H_i) \right)$

($\because$ $St(\mathfrak{X})$ comes from $G(A_k)$ and $\ker \partial x_i = \text{Image of } G(k)$)

Since G satisfies Hasse principle, $Ш^1(k, G) = \{1\}$. ($\because$ G is semi-simple, simply

For any $\xi \in Ш^1(k, H_i)$, ξ vanishes in $H^1(k, G)$.

$\Rightarrow$ $\exists$ $x \in X(k)$ such that $\partial_{x_i}(x) = \xi$.

On the other hand, $\prod_{v \in \Omega_k} \partial^v_{x_i}(x)$ is the trivial element. $\exists \sigma_A \in G(A_k)$ s.t $\sigma_A \cdot x_i = x$.

By strong approximation for G, one has $G(A_k) = G(k) \cdot St(\mathfrak{X})$.

$\Rightarrow$ $\sigma_A = g \cdot s_A$, where $s_A \in St(\mathfrak{X})$.

$\Rightarrow$ $g^{-1} x = s_A x_i \in \left(\prod_{v \in \Omega_k} \mathfrak{X}(O_{k_v}) \right)$

$\Rightarrow$ $g^{-1} x \in \mathfrak{X}(O_k) \cap St(\mathfrak{X}) x_i$.

- $\mathcal{X}(\mathcal{O}_{\mathbb{A}}) \cap St(\mathcal{X}) x_2 = \dot{\bigcup_j} \left(G(k) y_j^{(2)} \cap St(\mathcal{X}) x_2 \right)$

$$= \dot{\bigcup_j} \left(G(k) \cdot y_j^{(2)} \cap St(\mathcal{X}) y_j^{(2)} \right)$$

(3.5) F-orbits

Prop Let H_{2j} be the stabilizer of $y_j^{(2)}$ in G. Then,

$$\Gamma \backslash \left(G(k) y_j^{(2)} \cap St(\mathcal{X}) y_j^{(2)} \right) \xrightarrow{\cong} H_{2j}(k) \backslash H_{2j}(\mathbb{A}_k) / H_{2j}(\mathbb{A}_k) \cap St(\mathcal{X}).$$

proof) Let $x \in G(k) y_j^{(2)} \cap St(\mathcal{X}) y_j^{(2)}$. $\exists g \in G(k)$ and $s_{\mathbb{A}} \in St(\mathcal{X})$ s.t $x = g \cdot y_j^{(2)} = s_{\mathbb{A}} \cdot y_j^{(2)}$

$\Rightarrow g^{-1} s_{\mathbb{A}} \in H_{2j}(\mathbb{A}_k)$.

Send $x \mapsto g^{-1} s_{\mathbb{A}}$.

Need: $G(\mathbb{A}_k) = G(k) \cdot St(\mathcal{X})$ (Strong approximation for G)

(3.6) Equi-distribution property.

(*) Assumption: $\#\{ y \in \Gamma x : \|y\|_\infty \le T \} \sim \dfrac{\omega_{H_x}\left(H_x(k_\infty) / \Gamma_{H_x} \right)}{\omega_G \left(G(k_\infty) / \Gamma \right)} \cdot \omega_x \left(G(k_\infty) \cap X(k_\infty, T) \right)$

where H_x is the stabilizer of x in G.

and $\Gamma_{H_x} = \Gamma \cap H_x(k_\infty)$

(i) G is anisotropic torus (Similar to Dirichlet unit theorem, by Shyr)

(ii) G is almost k-simple and $G(k_\infty)$ is not compact and $X = G/H$ symmetric

$H = \{ g \in G : \sigma g = g \}$ where $\sigma : G \to G$ automorphism s.t $\sigma^2 = id$.

$k = \mathbb{Q}$:

k = number field : Benoist - oh

(iii) (E M S) Both G and H are connected reductive over $\mathbb{Q}$

and $\hat{G}(\mathbb{Q}) = \hat{H}(\mathbb{Q}) = \{1\}$. H is not contained in any proper parabolic $\mathbb{Q}$-subgp of G.

(3.7) Main Result.

Let $X = G/H$. In this case, one knows $\mathrm{Br}(X)/\mathrm{Br}_0(X) \cong \mathrm{Pic}(H)$ is finite.

Fix an injective homomorphism $\gamma : \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$. Then, any element $\xi \in \mathrm{Br}(X)$

can be viewed as a continuous fcn on $X(k_v)$ and $X(\mathbb{A}_k)$.

by $x_v \in X(k_v) \longrightarrow \mathrm{Inv}_v(\xi(x_v)) \xrightarrow{\gamma} \mathbb{C}^\times$

$(x_v) \in X(\mathbb{A}_k) \longrightarrow ev_\xi((x_v)) \in \mathbb{Q}/\mathbb{Z} \xrightarrow{\gamma} \mathbb{C}^\times$.

Def For any $\xi \in \mathrm{Br}(X)$, we define

$$N_v(\mathfrak{X}, \xi) = \int_{\mathfrak{X}(\mathcal{O}_{k_v})} \gamma \cdot \mathrm{Inv}_v \cdot \xi \quad \omega_X \qquad (v < +\infty)$$

$$N_\infty(X, T, \xi) = \int_{X(k_\infty, T)} \gamma \cdot \mathrm{Inv}_\infty \cdot \xi \quad \omega_X$$

Then, if the assumption $(*)$ holds, then

$$N(\mathfrak{X}, T) = \#\{x \in \mathfrak{X}(\mathcal{O}_k) : \|x\|_\infty \le T\}$$

$$\sim \sum_{\xi \in (\mathrm{Br}(X)/\mathrm{Br}_0(X))} \left(\prod_{v < +\infty} N_v(\mathfrak{X}, \xi) \cdot N_\infty(X, T, \xi) \right)$$

sketch of proof)

$$\mathfrak{X}(\mathcal{O}_k) = \dot{\bigcup_i} \left(\mathfrak{X}(\mathcal{O}_k) \cap St(\mathfrak{X}) x_i \right)$$

$$\mathfrak{X}(\mathcal{O}_k) \cap St(\mathfrak{X}) x_i = \bigcup_j \left(G(k) y_j^{(i)} \cap St(\mathfrak{X}) x_i \right)$$

$$= \dot{\bigcup_j} \left(G(k) y_j^{(i)} \cap St(\mathfrak{X}) y_j^{(i)} \right)$$

$$G(k) y_j^{(i)} \cap St(\mathfrak{X}) y_j^{(i)} = \dot{\bigcup_l} \Gamma \xi_l^{(i,j)}$$

Applying $(*)$, $\displaystyle N(\mathfrak{X}, T) \sim \sum_i \sum_j \sum_l \frac{\omega_{H_{i,j,l}}(H_{i,j,l}(k_\infty)/\Gamma_{i,j,l})}{\omega_G(G(k_\infty)/\Gamma)} \cdot \omega_X(G(k_\infty) \xi_l^{(i,j)} \cap X(k_\infty, T))$

$$\tau(H) \cdot \# Ш^1(k, H) = \# \mathrm{Pic}(H)$$

$$N(\mathfrak{X}, T) = \# \mathrm{Pic}(H) \int_{\left(\prod_{v < \infty_k} \mathfrak{X}(\mathcal{O}_{k_v}) \times X(k_\infty, T) \right)^{\mathrm{Br}(X)/\mathrm{Br}_0(X)}} \omega_X$$

§4. An example.

(4.1) Let $p(x)$ be an irreducible monic polynomial of deg ≥ 2 over $\mathbb{Z}$.

: $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, $a_1, \dots, a_n \in \mathbb{Z}$.

Question. $N(\mathfrak{X}, T) = \#\{(x_{ij}) \in M_n(\mathbb{Z}) : \det(\lambda I_n - (x_{ij})) = p(\lambda) \text{ and } \sqrt{\sum x_{ij}^2} \leq T\}$

- $\mathfrak{X}$: scheme over $\mathbb{Z}$ defined by the equation $\det(\lambda I_n - (x_{ij})) = p(\lambda)$. x_{ij} variables.

$$v = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ & & & & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix} \in \mathfrak{X}(\mathbb{Z}) \neq \phi .$$

- $X = \mathfrak{X} \times_{\mathbb{Z}} \mathbb{Q}$ a homogeneous space of $SL_n/\mathbb{Q}$, action : conjugation.
- Let $K = \mathbb{Q}(\theta)$, where θ is a root of $p(\lambda)$

Then, the stabilizer H of v = $\begin{cases} \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \\ \mathrm{Res}'_{K/\mathbb{Q}}(\mathbb{G}_m) \end{cases}$

Recall $N(\mathfrak{X}, T) \sim \sum_{\xi \in Br(X)/Br_0(X)} \prod_{p: \text{prime}} N_p(\mathfrak{X}, \xi) N_{\mathbb{R}}(X, \xi, T)$ as $T \to \infty$.

$Br(X)/Br_0(X) \cong Pic(S)$, where $S = \mathrm{Res}'_{K/\mathbb{Q}}(\mathbb{G}_m)$ $(\cong \mathbb{G}_m^{n-1})$

- By $H^p(\mathbb{Q}, H^q(\bar{S}, \mathbb{G}_m)) \Rightarrow H^{p+q}(S, \mathbb{G}_m)$

$$0 \to H^1(\mathbb{Q}, \bar{\mathbb{Q}}[S]^\times) \to H^1(S, \mathbb{G}_m) \to H^1(\bar{S}, \mathbb{G}_m)^{Gal(\bar{\mathbb{Q}}/\mathbb{Q})}$$

$\bar{\mathbb{Q}}[S]^\times = \bar{\mathbb{Q}}[\mathbb{G}_m^{n-1}]^\times$, $H^1(S,\mathbb{G}_m) = Pic(S)$, $H^1(\bar{S},\mathbb{G}_m) = Pic(\bar{S}) = 0$

Since $1 \to \mathrm{Res}'_{K/\mathbb{Q}}(\mathbb{G}_m) \to \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{\text{norm}} \mathbb{G}_m \to 1$,

$\Rightarrow 0 \to \mathbb{Z} \to \mathrm{Ind}_{Gal(L/K)}^{Gal(L/\mathbb{Q})} \mathbb{Z} \to \hat{S}_{\bar{\mathbb{Q}}} \to 0$ where L is Galois closure of $K/\mathbb{Q}$

and $\bar{\mathbb{Q}}[S]^\times = \hat{S} \times \bar{\mathbb{Q}}^\times$ (Rosenlicht)

$\rightsquigarrow Pic(S) = H^1(\mathbb{Q}, \hat{S}) \oplus H^1(\mathbb{Q}, \bar{\mathbb{Q}}^\times)$, with $H^1(\mathbb{Q}, \bar{\mathbb{Q}}^\times) = 0$

By taking Galois cohomology, one has

$$H^1(L/\mathbb{Q}, \mathrm{Ind}_{Gal(L/K)}^{Gal(L/\mathbb{Q})} \mathbb{Z}) \to H^1(L/\mathbb{Q}, \hat{S}) \to H^2(L/\mathbb{Q}, \mathbb{Z}) \to H^2(L/\mathbb{Q}, \mathrm{Ind}_{Gal(L/K)}^{Gal(L/\mathbb{Q})} \mathbb{Z})$$

Observe that $\begin{cases} H^1(L/\mathbb{Q}, \mathrm{Ind}_{L/K}^{L/\mathbb{Q}} \mathbb{Z}) = H^1(L/K, \mathbb{Z}) = 0 \\ H^2(L/\mathbb{Q}, \mathrm{Ind}_{L/K}^{L/\mathbb{Q}} \mathbb{Z}) = H^2(L/K, \mathbb{Z}) \end{cases}$ (Shapiro's lemma)

So, one has

$$0 \to H^1(L/\mathbb{Q}, \hat{S}) \to H^1(L/\mathbb{Q}, \mathbb{Z}) \to H^2(L/K, \mathbb{Z}) \quad \rightsquigarrow H^1(L/\mathbb{Q}, \hat{S}) = \ker(H^1(L/\mathbb{Q}, \mathbb{Z}) \to H^2(L/K, \mathbb{Z}))$$

By inflation-restriction sequence,

$$0 \to H^1(L/\mathbb{Q}, \hat{S}) \xrightarrow[\mathrm{Inf}]{\cong} H^1(\mathbb{Q}, \hat{S}) \xrightarrow[\mathrm{res}]{} H^1(L, \hat{S}) = \mathrm{Hom}(\mathrm{Gal}(\bar{\mathbb{Q}}/L), \mathbb{Q}/\mathbb{Z}) = 0 .$$

$$\Rightarrow \mathrm{Pic}(S) = H^1(\mathbb{Q}, \hat{S}) = H^1(L/\mathbb{Q}, \hat{S}).$$
$$= \ker(H^1(L/\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to H^1(L/K, \mathbb{Q}/\mathbb{Z}))$$
$$= \ker(\widehat{\mathrm{Gal}(L/\mathbb{Q})} \to \widehat{\mathrm{Gal}(L/K)})$$

Theorem $N(\mathfrak{X}, T) \sim \prod_{p:\text{prime}} N_p(\mathfrak{X}) \cdot N_{\mathbb{R}}(X, T)$ as $T \to \infty$.

proof). For any $\xi \in \mathrm{Br}(X)/\mathrm{Br}_0(X)$, $\xi \neq 1$, this is equivalent to say $\xi \in \widehat{\mathrm{Gal}(L/\mathbb{Q})}$, $\xi \neq 1$.

$\exists$ a cyclic extension A of $\mathbb{Q}$ such that $\mathrm{Gal}(A/\mathbb{Q}) = \langle \xi \rangle$

By Minkowski, $\exists\, p$ prime such that p ramified over $A/\mathbb{Q}$

$\Rightarrow \exists\, u_p \in \mathbb{Z}_p^\times$ such that $1 \neq \rho_p(u_p) \in \mathrm{Gal}(A_{\mathfrak{P}}/\mathbb{Q}_p) \subseteq \mathrm{Gal}(A/\mathbb{Q})$

local Artin map.

$\Rightarrow \xi(\rho_p(u_p)) \neq 1.$

Then, $$\int_{\mathfrak{X}(\mathbb{Z}_p)} \xi\, \omega_X = \int \xi(\rho_p(u_p))\, \xi\, \omega_X = \xi(\rho_p(u_p)) \int_{\mathfrak{X}(\mathbb{Z}_p)} \xi\, \omega_X$$

$\begin{pmatrix} u_p & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \leftarrow GL_n$

$\Rightarrow N_p(\mathfrak{X}, \xi) = 0.$

Remark There is an example. ξ play roles.

(4.3) Corollary (Eskin - Mozes - Shah)

If $p(\lambda)$ is split completely over $\mathbb{R}$, and $\mathbb{Z}[\theta]$ is the ring of integers of $K = \mathbb{Q}(\theta)$ where θ is a root of $p(\lambda)$, then

$$N(\mathfrak{X}, T) \sim \frac{2^{n-1} \cdot hR\omega_n}{\sqrt{D} \cdot \prod_{i=2}^{n} \Lambda(\frac{i}{2})} T^{\frac{1}{2}n(n-1)} \quad \text{as } T \to +\infty$$

where h is the class number of $\mathbb{Z}[\theta]$, R is the regulator, D is the discriminant of $p(\lambda)$, ω_n is the volume of unit ball in $\mathbb{R}^{\frac{1}{2}n(n-1)}$, $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$

proof of corollary : Since $\mathbb{Z}[\theta]$ is the ring of integers, one has $\mathbb{Z}[\theta]\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is the direct sum of discrete valuation rings.

$$1 \to (\mathbb{Z}[\theta]\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times \to GL_n(\mathbb{Z}_p) \to \mathcal{X}(\mathbb{Z}_p) \to 1$$

$$N_p(\mathcal{X}) = \int_{\mathcal{X}(\mathbb{Z}_p)} \omega_X = \frac{\mathrm{vol}(GL_n(\mathbb{Z}_p))}{\mathrm{vol}(\mathbb{Z}[\theta]\otimes_{\mathbb{Z}}\mathbb{Z}_p)}$$

$$1 \to SL_n(\mathbb{Z}_p) \to GL_n(\mathbb{Z}_p) \xrightarrow{\det} \mathbb{Z}_p^\times \to 1$$

$$\Rightarrow \mathrm{vol}(GL(\mathbb{Z}_p)) = (1-\tfrac{1}{p})\,\mathrm{vol}(SL_n(\mathbb{Z}_p))$$

$$\mathrm{vol}(\mathbb{Z}[\theta]\otimes_{\mathbb{Z}}\mathbb{Z}_p) = \prod_{\mathfrak{p}\mid p}\left(1-\frac{1}{N(\mathfrak{p})}\right)$$

$$\Rightarrow \prod_{p \text{ primes}} N_p(\mathcal{X}) = \frac{\zeta_K(s)}{\zeta(s)}\Big|_{s=1} \prod_{p \text{ primes}} \mathrm{vol}(SL_n(\mathbb{Z}_p))$$

$$\underset{\substack{\text{By class}\\ \text{number formula}}}{=} \frac{2^{n-1}\cdot hR}{\sqrt{D}} \prod_{i=2}^{n} \zeta(i)^{-1}$$

- over $\mathbb{R}$, one has $1 \to (\mathbb{R}^\times)^n \to GL_n(\mathbb{R}) \xrightarrow{\pi} X(\mathbb{R}) \to 1$

By Iwasawa decomposition, $GL_n(\mathbb{R}) = B\cdot O(n)$

Then, $1 \to \mathrm{diag}(\pm1,\pm1,\cdots,\pm1) \to B\times O(n) \to GL_n(\mathbb{R}) \to 1$

$\rightsquigarrow B = (\mathbb{R}^\times)^n \times N$, $N \cong \mathbb{R}^{\frac{1}{2}n(n-1)}$

Therefore,

$$\int_{X(\mathbb{R},T)} \omega_X = \frac{1}{2^n}\mathrm{vol}(O(n)\cap\pi^{-1}(X(\mathbb{R},T)))\cdot \mathrm{vol}(N\cap\pi^{-1}(X(\mathbb{R},T)))$$

Since $O(n)$ is compact,

$$\lim_{T\to\infty} \mathrm{vol}(O(n)\cap\pi^{-1}(X(\mathbb{R},T))) = \mathrm{vol}(O(n)) = 2^n\pi^{\frac{1}{2}n(n+1)}\cdot\prod_{i=1}^{n}\Gamma(\tfrac{i}{2})^{-1}$$

$$\lim_{T\to\infty} \mathrm{vol}(N\cap\pi^{-1}(X(\mathbb{R},T))) \sim w_n T^{\frac{1}{2}n(n-1)} \text{ as } T\to\infty.$$

$\boxed{\text{Problem}}$
- Remove conditions. $p(\lambda)$ splits completely in $\mathbb{R}$ and $\mathbb{Z}[\theta]$ is the ring of integers
- More challenging question: Work out the formula for general number field. $\mathbb{Q}(\sqrt{-1})$.