Postech lectures on the Brauer group and the Brauer–Manin obstruction

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Contents

*Notes based on the forthcoming book "The Brauer–Grothendieck group" by J.-L. Colliot-Thél'ene and A.N. Skorobogatov. Please email A.S. for any questions or comments.

1 Brauer groups of fields and schemes

1.1 Central simple algebras

Definition 1.1 *A finite-dimensional associative k-algebra A is called* **simple** *if the only two-sided ideals of A are 0 and A. A finite-dimensional associative k-algebra A is called* **central** *if its centre is k. A* **central simple algebra** *is a k-algebra that is both central and simple.*

Properties. 1. Any central division algebra is a central simple algebra. Example: quaternions.

2. For any integer $n \geq 1$ the algebra of matrices $M_n(k)$ is a central simple algebra. More generally, if D is a central division algebra, then $M_n(D)$ is a central simple algebra [GS17, Example 2.1.2].

3. $M_m(k) \otimes_k M_n(k) \cong M_{mn}(k)$.

Later we will use the following important property of matrix algebras.

Proposition 1.2 *Any automorphism of the k-algebra $M_n(k)$ is induced by conjugation by an invertible matrix. This invertible matrix is well defined up to multiplication by a scalar matrix.*

Proof. [GS17, Lemma 2.4.1, Cor. 2.4.2]. □

The structure of central simple algebras is described by a theorem of Wedderburn.

Theorem 1.3 (Wedderburn) *For any central simple algebra A there is a central division algebra D such that $A \cong D \otimes_k M_n(k) = M_n(D)$.*

The integer n is well defined, and the algebra D is well defined up to a non-unique isomorphism. Proofs of this fundamental theorem can be found in [BouVIII, §5, no. 4, Cor. 2], [GS17, Thm. 2.1.3].

Corollary 1.4 *Any central simple algebra over an algebraically closed field k is isomorphic to a matrix algebra $M_n(k)$.*

Proof. We need to prove that a central division k-algebra D coincides with its centre k. Pick any $x \in D$. Let $I \subset k[t]$ be the ideal consisting of polynomials vanishing on x. This is a non-zero ideal, generated by some $f(t) \in k[t]$. Since D is a division algebra, $f(t)$ is irreducible. As k is algebraically closed, $f(t)$ has degree 1, hence $x \in k$. □

Theorem 1.5 *Let k be a field and let A be a finite-dimensional k-algebra. Then A is a central simple algebra if and only if there exists a positive integer n and a finite field extension K/k such that $A \otimes_k K$ is isomorphic to $M_n(K)$. Moreover, if this is so, one can choose K separable over k.*

Proof. See [GS17, Thm. 2.2.1, Thm. 2.2.7] and [BouVIII, §10, no. 3, Prop. 4]. □

This theorem and properties 2 and 3 immediately imply that the tensor product $A \otimes_k B$ of two central simple algebras is again a central simple algebra. It also immediately implies that the dimension of a central simple algebra over its centre k is a square of a positive integer d. This integer d is called the *degree* of the algebra.

Two central simple algebras A and B are called *equivalent* if there are n and m such that $A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$. The relation is transitive by property 3. The equivalence class of k consists of the matrix algebras of all sizes.

Theorem 1.6 (Brauer) *The tensor product equips the set of equivalence classes of central simple algebras over k with the structure of an abelian group. It is called the* **Brauer group** *of k and is denoted by $\mathrm{Br}(k)$.*

Proof. The neutral element is the class of k. Associativity follows from the associativity of the tensor product. Commutativity follows from the isomorphisms $A \otimes B \xrightarrow{\sim} B \otimes A$ given by $x \otimes y \mapsto y \otimes x$. The inverse element of the class of A is the equivalence class of the *opposite* algebra A^{op}. Indeed, $A \otimes_k A^{\mathrm{op}}$ is a central simple algebra, and there is a non-zero homomorphism $A \otimes_k A^{\mathrm{op}} \to \mathrm{End}_k(A)$ that sends $a \otimes b$ to $x \mapsto axb$. It is injective since a central simple algebra has no two-sided ideals, and hence is an isomorphism by the dimension count. □

We write the group operation in $\mathrm{Br}(k)$ additively.

By Corollary 1.4, the Brauer group of an algebraically closed field is zero. By Theorem 1.5 this also holds for a separably closed field. Since $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only finite dimensional division $\mathbb{R}$-algebras (and $\mathbb{C}$ is not central), we see from Theorem 1.3 that $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2$.

Given a field extension K/k there is a natural *restriction* map

$$\mathrm{res}_{K/k} : \mathrm{Br}(k) \to \mathrm{Br}(K)$$

defined by $[A] \mapsto [A \otimes_k K]$. The kernel of $\mathrm{res}_{K/k}$ is denoted by $\mathrm{Br}(K/k)$ and is called the *relative Brauer group*.

1.2 Galois cohomology and Galois descent

Let G be a group that acts on a not necessarily commutative group A preserving its group structure. We denote the result of applying $\sigma \in G$ to $a \in A$ by ${}^{\sigma}a$. A

1-*cocycle* is a function $a = \{a_\sigma\} : G \to A$ which satisfies the relation

$$a_{\sigma\tau} = {}^\sigma a_\tau \cdot a_\sigma$$

for all $\sigma, \tau \in G$. The function $G \to A$ whose image is the identity element of A is called the trivial cocycle. Let $Z^1(G, A)$ be the set of 1-cocycles. Two cocycles $\{a_\sigma\}$ and $\{b_\sigma\}$ are called *equivalent* if there exists $c \in A$ such that for any $\sigma \in G$ one has

$$a_\sigma = {}^\sigma c \cdot b_\sigma \cdot c^{-1}.$$

The 1-*cohomology set* $\mathrm{H}^1(G, A)$ is defined as the set of equivalence classes of $Z^1(G, A)$ with respect to this relation. The class of the trivial cocycle is the *distinguished* point of $\mathrm{H}^1(G, A)$, so we can talk about $\mathrm{H}^1(G, A)$ as a pointed set.

Now suppose that G is a profinite group and the action of G on A is *continuous* when A is given the discrete topology. One defines the continuous cohomology pointed set $\mathrm{H}^1(G, A)$ as the direct limit of the pointed sets $\mathrm{H}^1(G/U, A^U)$, where $U \subset G$ ranges over all open normal subgroups – any such subgroup being of finite index in G. Alternatively, one defines $\mathrm{H}^1(G, A)$ as the set of equivalence classes of *continuous* cocycles $G \to A$.

An important particular case is when k is a field with separable closure k_s and absolute Galois group $\Gamma = \mathrm{Gal}(k_\mathrm{s}/k)$ acting on the group of k_s-points of an algebraic group A over k. The pointed set $\mathrm{H}^1(\Gamma, A(k_\mathrm{s}))$ does not depend on the choice of k_s; it is well defined up to canonical isomorphism and is denoted by $\mathrm{H}^1(k, A)$. The map $K \mapsto \mathrm{H}^1(K, A \times_k K)$ defines a functor from the category of field extensions of k to the category of pointed sets.

We shall mostly deal with the case of the projective linear group, so let us recall its definition. The multiplicative group $\mathbb{G}_{m,k}$ represents the functor associating to a commutative k-algebra R the group of invertible elements R^*. The algebraic group $\mathrm{GL}_{n,k}$ represents the functor $\mathrm{GL}_n(R)$. (In particular, $\mathbb{G}_{m,k} = \mathrm{GL}_{1,k}$.) Finally, the algebraic group $\mathrm{PGL}_{n,k}$ is defined by the exact sequence of algebraic k-groups

$$1 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathrm{GL}_{n,k} \longrightarrow \mathrm{PGL}_{n,k} \longrightarrow 1. \tag{1}$$

Let X be a variety over k. Let K/k be a Galois extension (not necessarily finite) with Galois group $\mathrm{Gal}(K/k)$. A k-variety Y is called a (K/k)*-form of* X if there is an isomorphism $Y \times_k K \cong X \times_k K$ of K-varieties. Using effectivity of Galois descent one shows that if X is a quasi-projective variety over k, then the (K/k)-forms of X are classified, up to isomorphism, by the elements of the Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{Aut}(X \times_k K))$ in such a way that the isomorphism class of X corresponds to the distinguished point. See [Po18, §4.4, §4.5] for a detailed proof of this classical result.

For example, the (k_s/k)-forms of a projective space are called *Severi–Brauer varieties.* It is not hard to see that Severi–Brauer varieties of dimension 1 are precisely plane projective conics. By a theorem of Châtelet, a Severi–Brauer variety is isomorphic to $\mathbb{P}_k^{n-1}$ if and only if it has a k-point. Note that the automorphism functor of $\mathbb{P}_k^{n-1}$ is represented by the group k-scheme $\mathrm{PGL}_{n,k}$.

More generally, suppose that we have a quasi-projective variety X over k endowed with an action of a group k-scheme A. By definition, each cohomology class in $\mathrm{H}^1(k, A)$ contains a 1-cocycle $c : \Gamma = \mathrm{Gal}(k_s/k) \to A(k_s)$; it comes from a 1-cocycle $c : \mathrm{Gal}(K/k) \to A(K)$ for some finite Galois extension $k \subset K$. The cocycle c defines a twisted action of $\mathrm{Gal}(K/k)$ on $X \times_k K$ as the composition of the action on $X \times_k K$ via the second factor with the action of $c(g) \in A(K)$. The cocycle condition is equivalent to this being an action of $\mathrm{Gal}(K/k)$ on $X \times_k K$ compatible with the action of $\mathrm{Gal}(K/k)$ on K by automorphisms. By effectivity of Galois descent, there exists a quasi-projective variety X^c over k such that the K-varieties $X \times_k K$ and $X^c \times_k K$ are isomorphic; this isomorphism identifies the action of $\mathrm{Gal}(K/k)$ on $X^c \times_k K$ via the second factor with the twisted by c action of $\mathrm{Gal}(K/k)$ on $X \times_k K$. The variety X^c is called the *twist* of X by c. By construction, it is a (k_s/k)-form of X. Replacing c by an equivalent cocycle gives rise to a variety non-canonically isomorphic to X^c. Particular cases of this situation include (see [Sko01, pp. 12–13], [Po18, §4.5]):

(a) Twists of the vector space k^n by a 1-cocycle with coefficients in $A = \mathrm{GL}_{n,k}$ are isomorphic to k^n, cf. [Po18, §1.3].

(b) Twists of the matrix algebra $M_n(k)$ by a 1-cocycle with coefficients in $A = \mathrm{PGL}_{n,k}$ are central simple algebras of degree n. Moreover, by [SerCL, Ch. X, §5, Prop. 8], this gives a bijection between the isomorphism classes of central simple algebras of degree n and the pointed set $\mathrm{H}^1(k, \mathrm{PGL}_{n,k})$.

(c) Torsors of an algebraic k-group A are obtained by twisting A by a 1-cocycle with coefficients in A acting on itself on the left. In this case A represents the automorphism functor of A considered together with its right action on itself, i.e. of A as a right A-torsor. Using effectivity of Galois descent one shows that the isomorphism classes of right A-torsors over k bijectively correspond to the elements of $\mathrm{H}^1(k, A)$. For example, the affine conic $x^2 - ay^2 = c$ is a torsor for the norm 1 torus given by $x^2 - ay^2 = 1$. Also, a smooth projective curve of genus 1 is a torsor for its Jacobian.

(d) Suppose that an algebraic k-group A acts on an algebraic k-group G by automorphisms. Twisting G by a 1-cocycle $\Gamma \to A$ one obtains a (k_s/k)-form of G. For example, the group of invertible elements of a central simple k-algebra of degree n is the group of k-points of a twist of $\mathrm{GL}_{n,k}$ by a 1-cocycle with values in $A = \mathrm{PGL}_{n,k}$. For any commutative algebraic group one defines quadratic twists by taking $A = \{\pm 1\}$, where -1 sends x to x^{-1}. For example, the quadratic twists of $\mathbb{G}_{m,k}$ are the norm tori $x^2 - ay^2 = 1$, where $a \in k^*$.

The quadratic twists of an elliptic curve $y^2 = x^3 + ax + b$ are the elliptic curves $cy^2 = x^3 + ax + b$, where $c \in k^*$.

Looking closer at the case of vector spaces one deduces the triviality of 1-cocycles with coefficients in $\mathrm{GL}_{n,k}$.

Theorem 1.7 (Speiser) *For any Galois extension of fields K/k with Galois group G we have $\mathrm{H}^1(G, \mathrm{GL}_n(K)) = \{1\}$.*

Proof. The automorphism functor of the n-dimensional vector space is represented by GL_n. The twist of k^n by a 1-cocycle $c : G \to \mathrm{GL}_n(K)$ is a vector space over k of dimension n, so it is isomorphic to k^n. This isomorphism, after tensoring with K, gives a linear transformation $\varphi \in \mathrm{GL}_n(K)$ such that $c(g) = {}^g\varphi \cdot \varphi^{-1}$. Thus c represents the trivial class. See also [GS17, Example 2.3.4] and [Po18, Prop. 1.3.15]. □

Theorem 1.8 (Hilbert's theorem 90) *For any Galois extension of fields K/k with Galois group G we have $\mathrm{H}^1(G, K^*) = 0$.*

This is a particular case of Speiser's theorem for $n = 1$.

Applying Hilbert's theorem 90 to (1) we see that for any field extension K/k the group of K-points of $\mathrm{PGL}_{n,k}$ is precisely $\mathrm{PGL}_n(K)$. Proposition 1.2 shows that the natural map

$$\mathrm{PGL}_n(K) \longrightarrow \mathrm{Aut}_{K-\mathrm{alg}}(M_n(K))$$

is an isomorphism of groups, where K-alg stands for the category of K-algebras. When K is a Galois extension of k, this isomorphism respects the Galois action on both sides. This confirms the fact alluded to above, namely, that the automorphism functor of the matrix algebra $M_n(k)$ is represented by the algebraic group $\mathrm{PGL}_{n,k}$.

Theorem 1.9 (Skolem–Noether) *All automorphisms of a central simple algebra over a field are inner automorphisms.*

Proof. Let A be a central simple algebra over a field k. Pick a finite Galois extension K/k that splits A. The homomorphism $A^* \to \mathrm{Aut}_{k-\mathrm{alg}}(A)$ sending an element to the conjugation by this element extends to a similar map over K. Let $G = \mathrm{Gal}(K/k)$. We then have the exact sequence of G-modules

$$1 \longrightarrow K^* \longrightarrow (A \otimes_k K)^* \longrightarrow \mathrm{Aut}_{K-\mathrm{alg}}(A \otimes_k K) \longrightarrow 1,$$

where surjectivity of the third map follows from Proposition 1.2. The long exact cohomology sequence gives an exact sequence of pointed sets

$$1 \longrightarrow k^* \longrightarrow A^* \longrightarrow \mathrm{Aut}_{k-\mathrm{alg}}(A) \longrightarrow \mathrm{H}^1(G, K^*).$$

Since $\mathrm{H}^1(G, K^*) = 0$ by Hilbert's theorem 90, the homomorphism $A^* \to \mathrm{Aut}_{k-\mathrm{alg}}(A)$ is surjective. □

1.3 Cohomological description of the Brauer group

Let K/k be a finite Galois extension of fields with Galois group G. Recall that a central simple algebra of degree n over k is split by K if and only if there exists an isomorphism of K-algebras $A \otimes_k K \cong M_n(K)$. Let us denote by $CSA_{n,K}$ the set of isomorphism classes of central simple algebras of degree n over k which are split by K. We have a bijection of pointed sets

$$CSA_{n,K} \tilde{\longrightarrow} \mathrm{H}^1(G, \mathrm{PGL}_n(K)).$$

Since $\mathrm{H}^1(G, \mathrm{GL}_n(K)) = \{1\}$ by Theorem 1.7, the exact sequence of pointed cohomology sets attached to (1)

$$\mathrm{H}^1(G, \mathrm{GL}_n(K)) \longrightarrow \mathrm{H}^1(G, \mathrm{PGL}_n(K)) \longrightarrow \mathrm{H}^2(G, K^*),$$

gives rise to maps

$$CSA_{n,K} \longrightarrow \mathrm{H}^2(G, K^*)$$

with trivial kernel. One easily checks that for given n and r there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & k^* & \to & \mathrm{GL}_n(k) & \to & \mathrm{PGL}_n(k) & \to & 1 \\ & & || & & \downarrow & & \downarrow & & \\ 1 & \to & k^* & \to & \mathrm{GL}_{nr}(k) & \to & \mathrm{PGL}_{nr}(k) & \to & 1 \end{array}$$

where the middle vertical map sends a matrix M to the matrix with r diagonal blocks equal to M and zero elsewhere. Replacing k by K and taking Galois cohomology we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{H}^1(G, \mathrm{PGL}_n(K)) & \to & \mathrm{H}^2(G, K^*) \\ \downarrow & & || \\ \mathrm{H}^1(G, \mathrm{PGL}_{nr}(K)) & \to & \mathrm{H}^2(G, K^*) \end{array}$$

The left vertical map can be identified with the map $CSA_{n,K} \to CSA_{nr,K}$ sending A to $A \otimes_k M_r(k)$. Passing to the limit over n we obtain a map of pointed sets

$$\mathrm{Br}(K/k) \longrightarrow \mathrm{H}^2(G, K^*)$$

with trivial kernel. Using Theorem 1.5 and passing to the limit over finite Galois extensions K/k, we get a map of pointed sets

$$\mathrm{Br}(k) \longrightarrow \mathrm{H}^2(k, k_{\mathrm{s}}^*)$$

with trivial kernel. One then establishes the following properties.

- These maps are homomorphisms of groups, hence they are injective. See [GS17, Prop. 2.7.9].

- These maps are surjective. This is proved by a cocycle computation using the classical construction of crossed products, see [SerCL, Ch. X, §5, Prop. 9]. See also [GS17, Thm. 4.4.1].

We summarise this as the following theorem.

Theorem 1.10 *For a field k and a Galois extension of fields K/k there are natural isomorphisms of abelian groups*

$$\mathrm{Br}(K/k) \tilde{\longrightarrow} \mathrm{H}^2(\mathrm{Gal}(K/k), K^*) \quad \text{and} \quad \mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{H}^2(k, k_{\mathrm{s}}^*).$$

The second isomorphism is functorial with respect to arbitrary field extensions of k, see [SerCL, Ch. 10, §4].

The cohomological description of the Brauer group is very useful. For example, it immediately gives

Corollary 1.11 *For any field k the Brauer group $\mathrm{Br}(k)$ is a torsion group.*

Proof. The group $\mathrm{Br}(k)$ is the direct limit of $\mathrm{Br}(K/k) = \mathrm{H}^2(\mathrm{Gal}(K/k), K^*)$, where K/k is a finite Galois extension. But if G is finite, then $\mathrm{H}^i(G, M)$, where M is any G-module and $i \geq 1$, is annihilated by the order of G. (This follows from the fact that the composition of the restriction to a subgroup $H \subset G$ followed by the corestriction is the multiplication by the index $[G : H]$. One applies this to the case when H is the identity element of G.) □

1.4 The Brauer–Azumaya group of a scheme

There are two ways to generalise the Brauer groups of fields to schemes. The definition of the Brauer group of a field k in terms of central simple algebras over k readily extends to schemes as the group of equivalence classes of Azumaya algebras. The following theorem is due to Azumaya, Auslander and Goldman, and Grothendieck, see [Gro68, I, Thm. 5.1] and [Mil80, Ch. IV, §2].

Theorem 1.12 *Let X be a noetherian scheme and let A be an $\mathcal{O}_X$-algebra which is an $\mathcal{O}_X$-module of finite type. The following conditions are equivalent:*

(i) *A is a locally free $\mathcal{O}_X$-module and for each $x \in X$ the fibre $A \otimes k(x)$ is a central simple algebra over the residue field $k(x)$.*

(ii) *A is a locally free $\mathcal{O}_X$-module and the natural homomorphism*

$$A \otimes_{\mathcal{O}_X} A^{\mathrm{op}} \longrightarrow \mathcal{E}nd_{\mathcal{O}_X-\mathrm{mod}}(A)$$

is an isomorphism.

(iii) *For each $x \in X$ there exist a positive integer r, a Zariski open set $U \subset X$ with $x \in U$ and a finite, surjective, étale morphism $U' \to U$ such that $A_{U'} \cong M_r(\mathcal{O}_{U'})$.*

(iv) *For each $x \in X$ there exist a positive integer r, a Zariski open set $U \subset X$ with $x \in U$ and a surjective étale morphism $U' \to U$ such that $A_{U'} \cong M_r(\mathcal{O}_{U'})$.*

An algebra A satisfying these equivalent conditions is called an *Azumaya algebra*. If X is connected, then the integer r in (iii) is constant on X. It is called the *degree* of the algebra.

A generalisation of the Skolem–Noether theorem leads to a proof that the set of isomorphism classes of Azumaya algebras of degree r on X is in a natural bijection with the étale Čech cohomology pointed set $\check{\mathrm{H}}^1_{\text{ét}}(X, \mathrm{PGL}_{r,X})$, see [Mil80, p. 122]. This pointed set classifies PGL_r-torsors on X [Mil80, Cor. III.4.7].

Two Azumaya algebras A and B on X are called *equivalent* if there exist locally free $\mathcal{O}_X$-modules P and Q locally of finite rank and an isomorphim of $\mathcal{O}_X$-algebras

$$A \otimes_{\mathcal{O}_X} \mathcal{E}nd(P) \cong B \otimes_{\mathcal{O}_X} \mathcal{E}nd(Q).$$

The set of equivalence classes is called the *Brauer–Azumaya group* $\mathrm{Br}_{Az}(X)$. Tensor product makes it into a commutative monoid such that the class of $\mathcal{O}_X$ is the identity element. It is actually an abelian group.

The group $\mathrm{Br}_{Az}(X)$ is a torsion group when X has finitely many connected components, which is the case when X is quasi-compact, see [Mil80, Prop. IV.2.7].

1.5 The Brauer–Grothendieck group of a scheme

The cohomological description of the Brauer group of a field also extends and gives rise to Grothendieck's definition of the (cohomological) Brauer group of a scheme X, which formally resembles his formula for the Picard group $\mathrm{Pic}(X) = \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{m,X})$.

Definition 1.13 *The Brauer–Grothendieck group of a scheme X is*

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X}).$$

For an affine scheme $X = \mathrm{Spec}(A)$, where A is a commutative ring, one often writes $\mathrm{Br}(A) := \mathrm{Br}(X)$.

In the particular case $X = \mathrm{Spec}(k)$, where k is a field, étale cohomology becomes Galois cohomology, so we obtain the classical description of the Brauer group of a field in terms of continuous 2-cocycles of its absolute Galois group $\Gamma = \mathrm{Gal}(k_\mathrm{s}/k)$, where k_s is a separable closure of k:

$$\mathrm{Br}(k) = \mathrm{H}^2(k, k_\mathrm{s}^*) = \mathrm{H}^2(\Gamma, k_\mathrm{s}^*).$$

A morphism of schemes $f : X \to Y$ which is locally of finite type gives rise to a morphism

$$f^* : \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y}) \longrightarrow \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X}). \qquad (2)$$

For $n = 2$ this gives a natural map of Brauer groups $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$, which is sometimes referred to as the *restriction* map. If K is a field and $M : \mathrm{Spec}(K) \to X$

is a K-point of X, then one writes $A(M) = M^*(A) \in \mathrm{Br}(K)$ and refers to $A(M)$ as the *value*, or specialisation, of A at M.

Let us fix an integer $n > 1$. The exact sequence (1) gives an exact sequence of group schemes over X

$$1 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \mathrm{GL}_{n,X} \longrightarrow \mathrm{PGL}_{n,X} \longrightarrow 1, \tag{3}$$

where $\mathbb{G}_{m,X} \to \mathrm{GL}_{n,X}$ is the central subgroup of scalar matrices. It gives rise to a boundary map of pointed cohomology sets

$$\delta_n : \check{\mathrm{H}}^1_{\text{ét}}(X, \mathrm{PGL}_{n,X}) \longrightarrow \check{\mathrm{H}}^2_{\text{ét}}(X, \mathbb{G}_m) \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) = \mathrm{Br}(X).$$

Theorem 1.14 (i) *The set* $\check{\mathrm{H}}^1_{\text{ét}}(X, \mathrm{PGL}_{n,X})$ *can be identified with the set of isomorphism classes of Azumaya algebras of degree* n *on* X.

(ii) *The boundary maps* δ_n *for* $n > 1$ *are compatible and induce a homomorphism of abelian groups* $\mathrm{Br}_{Az}(X) \to \mathrm{Br}(X)$.

(iii) *This homomorphism* $\mathrm{Br}_{Az}(X) \to \mathrm{Br}(X)$ *is injective.*

(iv) $\delta_n(\check{\mathrm{H}}^1_{\text{ét}}(X, \mathrm{PGL}_{n,X})) \subset \mathrm{Br}(X)[n]$.

Proof. See [Gro68, Prop. I.1.4] and [Mil80, Thm. IV.2.5]. Milne also gives a proof of (iii) via gerbes, which does not use the exact sequence (3). □

The fundamental result linking the Brauer–Azumaya group to the Brauer–Grothendieck group is the following theorem of Gabber. De Jong's proof of this theorem can be found in [deJ].

Theorem 1.15 (Gabber) *Let* X *be a quasi-compact separated scheme with an ample invertible sheaf, for example, a quasi-projective scheme over the spectrum of a commutative ring. Then the map*

$$\mathrm{Br}_{Az}(X) \longrightarrow \mathrm{Br}(X)_{\mathrm{tors}}$$

is an isomorphism.

By definition (see [Stacks, Def. 27.26.1]), X has an ample invertible sheaf means that X is quasi-compact, separated and there exists an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules with the following property: for each $x \in X$ there is an $s \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})$ for some $n \geq 1$ such that $s(x) \neq 0$ and the open subset $s \neq 0$ is affine.

2 Brauer group of a variety over a field

2.1 The Brauer group and cohomology with finite coefficients

The link of the Brauer group to étale cohomology with finite coefficients is provided by the Kummer exact sequence

$$1 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{x \mapsto x^{\ell^n}} \mathbb{G}_{m,X} \longrightarrow 1.$$

Here ℓ is a prime invertible on X and n is a positive integer. The associated long exact sequence of cohomology gives an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X)[\ell^n] \longrightarrow 0. \tag{4}$$

At the level of H^1 the Kummer sequence gives an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathbb{G}_m)/\mathrm{H}^0(X, \mathbb{G}_m)^{\ell^n} \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{Pic}(X)[\ell^n] \longrightarrow 0,$$

where $\mathrm{H}^0(X, \mathbb{G}_m)^{\ell^n}$ stands for the group of ℓ^n-powers of invertible regular functions on X. At the level of H^3 we have another useful exact sequence

$$0 \longrightarrow \mathrm{Br}(X)/\ell^n \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)[\ell^n] \longrightarrow 0. \tag{5}$$

2.2 The geometric Brauer group

Let X be a variety over a field k of characteristic exponent p. Write $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$, where k_{s} is a separable closure of k.

Definition 2.1 *The group* $\mathrm{Br}(X^{\mathrm{s}})$ *is called the* **geometric** *Brauer group of* X*. We denote by* $\mathrm{Br}^0(X^{\mathrm{s}})$ *the divisible subgroup of* $\mathrm{Br}(X^{\mathrm{s}})$*.*

Proposition 2.2 *Let X be a variety over a field k and let n be a positive integer coprime to* $\mathrm{char}(k)$*. Then the group* $\mathrm{Br}(X^{\mathrm{s}})[n]$ *is finite.*

Proof. The Kummer exact sequence (4) shows that $\mathrm{Br}(X^{\mathrm{s}})[n]$ is a quotient of $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_n)$, which is finite by [SGA4$\frac{1}{2}$, Finitude, Thm. 1.1]. □

Let ℓ be a prime, $\ell \neq p$. In this section we describe the ℓ-primary subgroups $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ and $\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\}$. Let us define the *Tate module* of $\mathrm{Br}(X^{\mathrm{s}})$ as

$$T_\ell \mathrm{Br}(X^{\mathrm{s}}) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{Br}(X^{\mathrm{s}})) = \varprojlim \mathrm{Br}(X^{\mathrm{s}})[\ell^n],$$

when $n \to \infty$. It is clear that $T_\ell \mathrm{Br}(X^{\mathrm{s}})$ is a torsion-free $\mathbb{Z}_\ell$-module. There are natural injective maps $T_\ell \mathrm{Br}(X^{\mathrm{s}})/\ell^n \hookrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n]$. By Nakayama's lemma,

$T_\ell \mathrm{Br}(X^{\mathrm{s}})$ is finitely generated, so is isomorphic to $\mathbb{Z}_\ell^r$ for some non-negative integer $r \leq \dim_{\mathbb{F}_\ell} \mathrm{Br}(X^{\mathrm{s}})[\ell]$. We have an isomorphism

$$T_\ell \mathrm{Br}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \tilde{\longrightarrow} \mathrm{Br}^0(X^{\mathrm{s}})\{\ell\}. \tag{6}$$

Let X be a smooth, proper, geometrically integral variety over k. Let $b_n = \dim \mathrm{H}^n_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell)$ be the n-th ℓ-adic *Betti number* of X^{s}. It is independent of ℓ and is equal to $\dim \mathrm{H}^n(X_{\mathbb{C}}, \mathbb{Q})$ when $k^{\mathrm{s}} \subset \mathbb{C}$. The *Picard number* ρ of X^{s} is the rank of the Néron–Severi group $\mathrm{NS}\,(X^{\mathrm{s}}) = \mathrm{NS}\,(\overline{X})$.

Proposition 2.3 *Let X be a smooth, proper, geometrically integral variety over a field k of characteristic exponent p. Then the following statements hold.*

(i) *For a prime $\ell \neq p$ there is an exact sequence of Γ-modules*

$$0 \longrightarrow \mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})\{\ell\} \longrightarrow \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0, \tag{7}$$

where

$$\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} = \big(\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))/(\mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell)\big) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}.$$

(ii) *If char$(k) = 0$, there is an exact sequence of Γ-modules*

$$0 \longrightarrow \mathrm{Br}^0(\overline{X}) \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \bigoplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0, \tag{8}$$

where $\mathrm{Br}^0(\overline{X}) \cong (\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$*; the direct sum is a finite abelian group.*

(iii) *When $k \subset \mathbb{C}$, the finite group $\bigoplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$ is isomorphic to the torsion subgroup of* $\mathrm{H}^3(X(\mathbb{C}), \mathbb{Z})$.

Proof. (i) Replacing X by X^{s} in the exact sequence (4) obtained from the Kummer sequence, gives the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n] \longrightarrow 0.$$

The neutral connected component of the Picard scheme $\mathbf{Pic}^0_{X/k}$ is a connected projective algebraic group over k, hence $A = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ is an abelian variety. Since $\ell \neq p$, the multiplication by ℓ map $A \to A$ is finite étale, hence it is surjective on k_{s}-points. Thus $\mathrm{Pic}(X^{\mathrm{s}}) = \mathbf{Pic}^0_{X/k}(k_{\mathrm{s}})$ is divisible by ℓ, so we can rewrite the previous exact sequence as follows:

$$0 \longrightarrow \mathrm{NS}\,(X^{\mathrm{s}})/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n] \longrightarrow 0. \tag{9}$$

By the finiteness of étale cohomology with finite coefficients [SGA4$\frac{1}{2}$, Finitude, Thm. 1.1], (9) is an exact sequence of *finite* abelian groups. Thus passing to the limit for $n \to \infty$ the sequence we obtain is still exact:

$$0 \longrightarrow \mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}_\ell} \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell \mathrm{Br}(X^{\mathrm{s}}) \longrightarrow 0, \tag{10}$$

where the second arrow is the definition of the ℓ-adic *cycle class map* cl_ℓ. Since $T_\ell \mathrm{Br}(X^{\mathrm{s}})$ is a free $\mathbb{Z}_\ell$-module, the ℓ-primary torsion subgroup $\mathrm{NS}\,(X^{\mathrm{s}})\{\ell\}$ is canonically isomorphic to $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$. We obtain an isomorphism of abelian groups $T_\ell \mathrm{Br}(X^{\mathrm{s}}) \cong \mathbb{Z}_\ell^{b_2-\rho}$, which, in view of the canonical isomorphism (6), gives

$$\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}.$$

If we repeat the same arguments at the level of H^3, we see that the Kummer sequence identifies $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}/\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\}$ with the kernel of the map

$$\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m).$$

Since the Tate module is torsion-free and the Brauer group is torsion, we get an isomorphism

$$\mathrm{Br}(X^{\mathrm{s}})\{\ell\}/\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} \tilde{\longrightarrow} \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)_{\mathrm{tors}}.$$

(ii) For an arbitrary separably closed field k_{s} a theorem of Gabber says that for almost all ℓ the group $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ is torsion-free. If k has characteristic 0, this is also a consequence of the comparison theorem between étale cohomology and classical Betti cohomology, see [Mil80, Thm. III.3.12].

(iii) Since the étale cohomology groups of a scheme over k^{s} with coefficients in a torsion sheaf of order coprime to $\mathrm{char}(k)$ do not change under extension of k^{s} to a bigger separably closed field [Mil80, Cor. VI.4.3], in the case $k^{\mathrm{s}} \subset \mathbb{C}$ we have $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) = \mathrm{H}^3_{\text{ét}}(X \times_k \mathbb{C}, \mathbb{Z}_\ell(1))$. The comparison theorem [Mil80, Thm. III.3.12] says that the latter group is isomorphic to the Betti cohomology group $\mathrm{H}^3(X \times_k \mathbb{C}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1)$. $\square$

2.3 Algebraic and transcendental Brauer groups

For a variety X over a field k there is a natural filtration on the Brauer group

$$\mathrm{Br}_0(X) \subset \mathrm{Br}_1(X) \subset \mathrm{Br}(X),$$

which is defined as follows.

Definition 2.4 *Let*

$$\mathrm{Br}_0(X) = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(X)], \quad \mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})].$$

The subgroup $\mathrm{Br}_1(X) \subset \mathrm{Br}(X)$ *is called the* **algebraic** *Brauer group of* X*, and the quotient* $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *is called the* **transcendental** *Brauer group of* X*.*

The Leray spectral sequence for the structure morphism $X \to \mathrm{Spec}(k)$ is

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m). \qquad (11)$$

It gives rise to the functorial exact sequence of terms of low degree

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1(k, k_{\mathrm{s}}[X]^*) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^\Gamma \longrightarrow \mathrm{H}^2(k, k_{\mathrm{s}}[X]^*) \\ \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}[X]^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned} \tag{12}$$

Proposition 2.5 *Let X be a variety over a field k such that $k_{\mathrm{s}}[X]^* = k_{\mathrm{s}}^*$ (for example, a proper and geometrically integral variety, or $\mathbb{A}^n_k$). Then there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^\Gamma \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X) \\ \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned} \tag{13}$$

This sequence is contravariant functorial in X.

Proof. This follows from (12), since by Hilbert's theorem 90 we have $\mathrm{H}^1(k, k_{\mathrm{s}}^*) = 0$. □

Remark If X has a k-point or, more generally, if X has a 0-cycle of degree 1, then each of the maps $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$ and $\mathrm{H}^3(k, k_{\mathrm{s}}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)$ in (13) has a retraction, hence is injective. (Then $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathrm{s}})^\Gamma$ is an isomorphism.) Indeed, a k-point on X defines a section of the structure morphism $X \to \mathrm{Spec}(k)$. A standard restriction-corestriction argument reduces the case when X has a 0-cycle of degree 1 to the case when X has a k-point.

In general, the quotient $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ can be computed as follows.

Proposition 2.6 *For each $n \geq 0$ the differential*

$$\mathrm{H}^n(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{H}^{n+2}(k, k_{\mathrm{s}}[X]^*) \tag{14}$$

from the spectral sequence (11) *coincides, up to sign, with the connecting map defined by the* 2*-extension of* Γ*-modules*

$$0 \longrightarrow k_{\mathrm{s}}[X]^* \longrightarrow k_{\mathrm{s}}(X)^* \longrightarrow \mathrm{Div}(X^{\mathrm{s}}) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \longrightarrow 0. \tag{15}$$

Remark The differential (14) can be seen as the map attached to the exact triangle

$$p_*\mathbb{G}_{m,X} \longrightarrow \tau_{[0,1]}\mathbf{R}p_*\mathbb{G}_{m,X} \longrightarrow (R^1p_*)\mathbb{G}_{m,X}[-1]$$

in the bounded below derived category $\mathcal{D}(k)$ of Γ-modules. Here $p : X \to \mathrm{Spec}(k)$ is the structure morphism, $\mathbf{R}p_* : \mathcal{D}(X) \to \mathcal{D}(k)$ is the derived functor from the bounded below derived category $\mathcal{D}(X)$ of étale sheaves on X to $\mathcal{D}(k)$, and $\tau_{[0,1]}$ is the truncation functor. Proposition 2.6 then follows from the fact that $\tau_{[0,1]}\mathbf{R}p_*\mathbb{G}_{m,X}$ is represented by the 2-term complex $k_{\mathrm{s}}(X)^* \to \mathrm{Div}(X^{\mathrm{s}})$ (Borovoi–van Hamel).

The spectral sequence (11) gives rise to a *complex*

$$\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})).$$

Assume $k_{\mathrm{s}}^* = k_{\mathrm{s}}[X]^*$. From the general structure of spectral sequences we see that if $\mathrm{H}^3(k, k_{\mathrm{s}}^*) = 0$ or if X has a k-point (or a 0-cycle of degree 1), then the above complex becomes an exact sequence

$$0 \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})).$$

Thus $\mathrm{Br}(X)/\mathrm{Br}_1(X) = \mathrm{Ker}(\beta)$. For concrete calculations of the Brauer group one would like to be able to compute the map β. As an approximation to this, we now describe the following composition:

$$\mathrm{Br}^0(X^{\mathrm{s}})^{\Gamma} \hookrightarrow \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{H}^2(k, N(X^{\mathrm{s}})), \qquad (16)$$

where $N(X^{\mathrm{s}})$ is the quotient of the Néron–Severi group $\mathrm{NS}\,(X^{\mathrm{s}})$ by its torsion subgroup. By the results of Section 2.2, this map coincides with β when k has characteristic 0, $\mathrm{H}^1(X, \mathcal{O}) = 0$, and the groups $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ and $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ are torsion-free for all primes ℓ, so our description covers many important cases. For the sake of simplicity we state the result in the case when X is a surface, referring to [CTS13, Prop. 4.1] for the general case.

Let X be a smooth, projective, geometrically integral surface over a field k of characteristic 0. Assume that k is a finitely generated subfield of $\mathbb{C}$. We have seen that the Néron–Severi group does not change when a separably closed ground field is extended to a larger separably closed field, hence we have an isomorphism $N(X^{\mathrm{s}}) \tilde{\longrightarrow} N(X_{\mathbb{C}})$. Let us write $\mathrm{H}^2(X_{\mathbb{C}})$ for the quotient of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ by its torsion subgroup. For a surface X the Poincaré duality gives rise to a perfect (unimodular) pairing

$$\mathrm{H}^2(X_{\mathbb{C}}) \times \mathrm{H}^2(X_{\mathbb{C}}) \longrightarrow \mathbb{Z}$$

given by the cup-product. By the Hodge index theorem, the restriction of this pairing to $N(X_{\mathbb{C}})$ has a non-zero discriminant. A classical argument based on the exponential exact sequence shows that $N(X_{\mathbb{C}})$ is a saturated subgroup of $\mathrm{H}^2(X_{\mathbb{C}})$, in the sense that the quotient is torsion-free.

Let $T(X_{\mathbb{C}})$ be the *lattice of transcendental cycles* of $X_{\mathbb{C}}$ defined as the orthogonal complement to $N(X_{\mathbb{C}})$ in $\mathrm{H}^2(X_{\mathbb{C}})$ with respect to the cup-product pairing. Thus $T(X_{\mathbb{C}})$ is a saturated subgroup of $\mathrm{H}^2(X_{\mathbb{C}})$, and $N(X_{\mathbb{C}}) \cap T(X_{\mathbb{C}}) = 0$. Write

$$N(X_{\mathbb{C}})^* = \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}), \qquad T(X_{\mathbb{C}})^* = \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z}).$$

The cup-product gives rise to the injective maps

$$N(X_{\mathbb{C}}) \hookrightarrow N(X_{\mathbb{C}})^*, \qquad T(X_{\mathbb{C}}) \hookrightarrow T(X_{\mathbb{C}})^*.$$

By the unimodularity of the pairing on $\mathrm{H}^2(X_{\mathbb{C}})$ we have canonical isomorphisms of finite abelian groups

$$N(X_{\mathbb{C}})^*/N(X_{\mathbb{C}}) = \mathrm{H}^2(X_{\mathbb{C}})/(N(X_{\mathbb{C}}) \oplus T(X_{\mathbb{C}})) = T(X_{\mathbb{C}})^*/T(X_{\mathbb{C}}).$$

We deduce a natural exact sequence

$$0 \longrightarrow N(X^{\mathrm{s}}) \longrightarrow N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

By the comparison theorem between classical and étale cohomology we have an isomorphism $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Z}_\ell \cong \mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1))$, compatible with the cycle class map and the cup-product, for any prime ℓ. Thus $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is the orthogonal complement to $\mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$ in $\mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1))$. In particular, $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is naturally a Γ-module, so that the previous 4-term exact sequence is an exact sequence of Γ-modules.

Since $N(X_{\mathbb{C}})$ is the orthogonal complement to $T(X_{\mathbb{C}})$ in $\mathrm{H}^2(X_{\mathbb{C}})$, we obtain $T(X_{\mathbb{C}})^* = \mathrm{H}^2(X_{\mathbb{C}})/N(X_{\mathbb{C}})$. Tensoring with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ we get

$$\mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \big(\mathrm{H}^2(X_{\mathbb{C}})/N(X_{\mathbb{C}})\big) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = \frac{\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))}{\mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

From the description of $\mathrm{Br}^0(X^{\mathrm{s}})$ given in Proposition 2.3 (i) we now obtain a canonical isomorphism of Γ-modules

$$\mathrm{Br}^0(X^{\mathrm{s}}) = \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z})$$

and an exact sequence of Γ-modules

$$0 \longrightarrow N(X^{\mathrm{s}}) \longrightarrow N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Br}^0(\overline{X}) \longrightarrow 0. \qquad (17)$$

The following proposition formally resembles Proposition 2.6.

Proposition 2.7 *Let X be a smooth, projective, geometrically integral surface over a field k of characteristic 0. The composed map (16) coincides, up to sign, with the connecting map*

$$\mathrm{Br}^0(\overline{X})^\Gamma \longrightarrow \mathrm{H}^2(k, N(X^{\mathrm{s}}))$$

defined by the 2-extension of Γ-modules (17).

Proof. See [CTS13, Prop. 4.1]. □

Remark This remark is a continuation of the previous remark and uses the same notation. Let X be a smooth, projective, geometrically integral surface over a subfield of $\mathbb{C}$ such that $\mathrm{Pic}(X^{\mathrm{s}})$ is torsion-free. Then the 2-term complex

$$N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z},$$

which is the middle part of (17), represents $\tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X}[1]$ in the bounded below derived category of Γ-modules. This explains the previous proposition, because the relevant differential in the spectral sequence coincides (up to sign) with the map attached to the exact triangle

$$(R^1p_*)\mathbb{G}_{m,X}[-1] \longrightarrow \tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X} \longrightarrow (R^2p_*)\mathbb{G}_{m,X}[-2].$$

See [GS, Prop. 1.2] for details.

It is known that the transcendental Brauer group $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ has finite index in $\mathrm{Br}(X^{\mathrm{s}})^\Gamma$, at least when the characteristic of the ground field k is 0, see [CTS13].

3 The Brauer–Manin set

3.1 Brauer groups of local and global fields

Let k be a number field, that is, a finite extension of $\mathbb{Q}$. Write Ω for the set of places of k. Let k_v be the completion of k at $v \in \Omega$. If v is a finite place, we denote by $\mathcal{O}_v$ the ring of integers of k_v.

Azumaya's theorem says that if R is a henselian local ring with residue field k, then the embedding of the closed point $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ induces an isomorphism $\mathrm{Br}(R)\tilde{\longrightarrow}\mathrm{Br}(k)$. The Brauer group of a finite field is zero, so $\mathrm{Br}(\mathcal{O}_v) = 0$.

Now let R be a henselian discrete valuation ring with fraction field K and *perfect* residue field k (e.g. $R = \mathcal{O}_v$). We have inclusions of discretely valued fields $K \subset K_{\mathrm{nr}} \subset K_{\mathrm{s}}$. The residue field of K_{nr} and K_{s} is the algebraic closure of k. We have $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K) = \mathrm{Gal}(k_{\mathrm{s}}/k)$.

By a theorem of Lang we have $\mathrm{Br}(K_{\mathrm{nr}}) = 0$. By Hilbert's theorem 90 the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q(K_{\mathrm{nr}}, K_{\mathrm{s}}^*)) \Rightarrow \mathrm{H}^{p+q}(K, K_{\mathrm{s}}^*) \qquad (18)$$

gives an isomorphism $\mathrm{H}^2(\Gamma, K_{\mathrm{nr}}^*)\tilde{\longrightarrow}\mathrm{Br}(K)$. Composing it with the Galois equivariant map $v : K_{\mathrm{nr}}^* \to \mathbb{Z}$ given by the valuation we obtain

$$\mathrm{Br}(K)\tilde{\longleftarrow}\mathrm{H}^2(\Gamma, K_{\mathrm{nr}}^*) \xrightarrow{v_*} \mathrm{H}^2(\Gamma, \mathbb{Z})\tilde{\longleftarrow}\mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Q}/\mathbb{Z}),$$

where the isomorphism $\mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z})\tilde{\longrightarrow}\mathrm{H}^2(\Gamma, \mathbb{Z})$ comes from Galois cohomology of the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \qquad (19)$$

The resulting map $\mathrm{Br}(K) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is called the *Witt residue*.

The choice of a generator of the maximal ideal of R splits the exact sequence

$$0 \longrightarrow R_{\mathrm{nr}}^* \longrightarrow K_{\mathrm{nr}}^* \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The Hochschild–Serre spectral sequence [Mil80, Ch. III, Thm. 2.20, Rem. 2.21]

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q(R_{\mathrm{nr}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}(R, \mathbb{G}_m),$$

in view of $\mathrm{H}^1(R_{\mathrm{nr}}, \mathbb{G}_m) = 0$, $\mathrm{H}^2(R_{\mathrm{nr}}, \mathbb{G}_m) = 0$ (by Azumaya's theorem), gives $\mathrm{H}^2(\Gamma, R^*_{\mathrm{nr}}) = \mathrm{Br}(R) = \mathrm{Br}(k)$, so we obtain that v_* is an isomorphism if k is finite. So in this case the Witt residue is an isomorphism.

Definition 3.1 *For each place v of k define*

$$\mathrm{inv}_v : \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

as follows. If v is finite, let inv_v be the Witt residue $\mathrm{Br}(k_v) \to \mathrm{H}^1(\mathbb{F}_v, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), \mathbb{Q}/\mathbb{Z})$ followed by the evaluation at the Frobenius element. If v is real, define inv_v by $\mathrm{Br}(k_v) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$. For a complex place v set $\mathrm{inv}_v = 0$.

Theorem 3.2 (i) *For each finite place v of k, the map inv_v is an isomorphism. For each real place v, the map inv_v is the injection $\mathrm{Br}(k_v) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$. For each complex place v we have $\mathrm{Br}(k_v) = 0$.*

(ii) *The diagonal map $\mathrm{Br}(k) \to \prod_{v\in\Omega} \mathrm{Br}(k_v)$ factors through the direct sum $\oplus_{v\in\Omega}\mathrm{Br}(k_v)$.*

(iii) *The maps inv_v fit into an exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{20}$$

where the map to $\mathbb{Q}/\mathbb{Z}$ is the sum of inv_v for all $v \in \Omega$.

The fact that (20) is a complex is a generalisation of the Gauss quadratic reciprocity law. Injectivity on the second arrow is a celebrated theorem of H. Hasse, R. Brauer and E. Noether, generalising results of Legendre and Hilbert.

3.2 Adèles and adelic points

We shall write S for a finite set of places of k containing all the archimedean places. Let $\mathcal{O}_k$ be the ring of integers of k and let $\mathcal{O}_{k,S}$ be the ring of S-integers, i.e. the elements of k that belong to $\mathcal{O}_v$ for $v \notin S$.

The product $\prod_{v\in\Omega} k_v$ is a topological ring equipped with the product topology, where each k_v carries its natural archimedean or non-archimedean topology. The ring of adèles $\mathbf{A}_k$ is defined as a subring of $\prod_{v\in\Omega} k_v$ given by the condition that all but finitely many components are in $\mathcal{O}_v$. The topology of $\mathbf{A}_k$ induced by the topology of $\prod_{v\in\Omega} k_v$ is such that a base is given by the open sets $\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v$, where U_v is open in k_v. We put

$$\mathbf{A}_{k,S} = \prod_{v\in S} k_v \times \prod_{v\notin S} \mathcal{O}_v.$$

Then $\mathbf{A}_k$ is the direct limit of the open subrings $\mathbf{A}_{k,S}$ over all finite $S \subset \Omega$ containing the archimedean places. We note that k is discrete in $\mathbf{A}_k$, and $\mathcal{O}_{k,S}$ is discrete in $\mathbf{A}_{k,S}$.

If $X \subset \mathbb{A}^n_k$ is an affine variety, then the set $X(\mathbf{A}_k)$ is identified with the closed subset of $\mathbf{A}_k^n$ and so acquires a locally compact Hausdorff subspace topology. This topology does not depend on the closed immersion $X \hookrightarrow \mathbb{A}^n_k$.

Let X be a variety over k (that is, a separated scheme of finite type over k). By [EGA, IV_3, §8.8] for some finite set T of places there exists a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{k,T}$ with generic fibre X. Let $S \subset \Omega$ be a finite set containing T. It is clear that an $\mathbf{A}_{k,S}$-valued point of $\mathcal{X}$ gives rise to an $\mathbf{A}_k$-valued point of $\mathcal{X} \times_{\mathcal{O}_{k,T}} \mathbf{A}_k$. Since $\mathcal{O}_{k,T} \subset k \subset \mathbf{A}_k$, we have $\mathcal{X} \times_{\mathcal{O}_{k,T}} \mathbf{A}_k = X \times_k \mathbf{A}_k$, so an $\mathbf{A}_k$-valued point of $\mathcal{X} \times_{\mathcal{O}_{k,T}} \mathbf{A}_k$ is identified with an $\mathbf{A}_k$-valued point of X. This gives rise to a map of sets

$$\varinjlim \mathcal{X}(\mathbf{A}_{k,S}) \longrightarrow \mathcal{X}(\mathbf{A}_k) = X(\mathbf{A}_k). \tag{21}$$

Here the limit is over S, and it does not depend on T. An $\mathbf{A}_k$-valued point of X comes from an $\mathbf{A}_{k,S}$-valued point of $\mathcal{X}$ for some S, so this map is bijective.

The following natural map of sets is a bijection:

$$\mathcal{X}(\mathbf{A}_{k,S}) \tilde{\longrightarrow} \prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v).$$

(This requires proving that for a quasi-compact and quasi-separated $\mathcal{X}$ over a ring A and a family of local A-algebras R_i, we have $\mathcal{X}(\prod R_i) = \prod \mathcal{X}(R_i)$, which is hard!) This implies that $X(\mathbf{A}_k)$ is the restricted topological product of $X(k_v)$, for $v \in \Omega$, with respect to $\mathcal{X}(\mathcal{O}_v)$, $v \notin S$. The sets $X(k_v)$ and $\mathcal{X}(\mathcal{O}_v)$ are topologised via gluing of affine set. This makes $\mathcal{X}(\mathbf{A}_{k,S})$ a locally compact Hausdorff topological space. If $S \subset S'$, then $\mathcal{X}(\mathbf{A}_{k,S}) \to \mathcal{X}(\mathbf{A}_{k,S'})$ is an open embedding. Using (21) we make $X(\mathbf{A}_k)$ a topological space in such a way that a subset of $X(\mathbf{A}_k)$ is open if its intersection with each $\mathcal{X}(\mathbf{A}_{k,S})$ is open. Then $X(\mathbf{A}_k)$ is a locally compact Hausdorff topological space with a countable basis of open sets. We note that the sets $\mathcal{X}(\mathbf{A}_{k,S})$ form an open covering of $X(\mathbf{A}_k)$. A morphism $f : X \to Y$ of varieties over k gives rise to a continuous map $X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$.

We refer to $X(\mathbf{A}_k)$ as the *adelic space* of X and call its elements the *adelic points* of X. If X is an affine variety over k, the topology of the adelic space $X(\mathbf{A}_k)$ is the natural topology defined earlier in the affine case. If X is proper, we can choose $\mathcal{X}$ proper over $\mathcal{O}_T$. For $v \notin S$, by the valuative criterion of properness, we have $X(k_v) = \mathcal{X}(\mathcal{O}_v)$, hence $X(\mathbf{A}_k)$ coincides with the product topological space $\prod_{v \in \Omega} X(k_v)$, and so is compact.

3.3 Evaluation maps

Let us discuss the evaluation map over a local field. Let X be a scheme over a field k and let $P \in X(k)$ be a k-point. For any $A \in \mathrm{Br}(X)$ with $A(P) = 0 \in \mathrm{Br}(k)$

there exists an étale map $f : U \to X$ such that $f^*A = 0 \in \mathrm{Br}(U)$ and there exists a k-point $M \in U(k)$ such that $f(M) = P$. Indeed, let R be the henselisation of the local ring of X at P. By Azumaya's theorem the image of A under the natural map $\mathrm{Br}(X) \to \mathrm{Br}(R)$ is zero. The ring R is a filtering direct limit of rings R_i, each of them equipped with an étale map $f_i : \mathrm{Spec}(R_i) \to X$ and a k-point M_i such that $f_i(M_i) = P$. The group $\mathrm{Br}(R)$ is the direct limit of the groups $\mathrm{Br}(R_i)$ [Mil80, Lemma I.1.16, Remark I.1.17 (a)] (which refers to SGA4, VII.5.8). Thus α goes to zero in $\mathrm{Br}(R_i)$ for some i, so we can take $U = \mathrm{Spec}(R_i)$.

If k is $\mathbb{R}$ or a p-adic field, then $f(U(k))$ is an open neighbourhood of P in the topology of k (by the implicit function theorem). Thus the evaluation map $ev_A : X(k) \to \mathrm{Br}(k_v)$ is locally constant.

Let us now discuss the adelic evaluation map.

Lemma 3.3 *Let k be a number field, let X be a variety over k and let $A \in \mathrm{Br}(X)$.*

(i) *There exist a finite set of places T, a morphism $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_T)$ with generic fibre X, and an element $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$ with image $A \in \mathrm{Br}(X)$.*

(ii) *For $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_T)$ as in (i), for any finite place $v \notin T$ and for any point $M_v \in \mathcal{X}(\mathcal{O}_v) \subset X(k_v)$ we have $A(M_v) = 0$.*

(iii) *If X is proper, there exists a finite set of places T such that for all $v \notin T$ and for any $M_v \in X(k_v)$ we have $A(M_v) = 0$.*

(iv) *The map*

$$ev_A : X(\mathbf{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which sends an adelic point $\{M_v\}$ to the sum $\sum_{v\in\Omega} \mathrm{inv}_v A(M_v) \in \mathbb{Q}/\mathbb{Z}$ is a well-defined continuous map whose image is annihilated by a positive integer.

Proof. (i) We have $X = \varprojlim \mathcal{X}_S$, where the limit is over the finite sets of places S. Since étale cohomology commute with limits, we have $\mathrm{Br}(X) = \varinjlim \mathrm{Br}(\mathcal{X}_S)$, which implies (i).

(ii) This follows from $\mathrm{Br}(\mathcal{O}_v) = 0$ (Azumaya's theorem).

(iii) If X is proper, then in (i) we can take $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_T)$ to be proper. Then for any finite place $v \notin S$ we have $\mathcal{X}(\mathcal{O}_v) = X(k_v)$. Now (iii) follows from (ii).

(iv) Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_T)$ be as in (i). Open sets

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v),$$

where S is a finite set of places of k containing T and U_v is an open set in $X(k_v)$ for $v \in S$, form a basis of open sets of $X(\mathbf{A}_k)$. By (ii), the map ev_A is well-defined on such open sets. It is continuous on each of these open sets. This follows from the fact that the evaluation maps $ev_A : \mathcal{X}(\mathcal{O}_v) \to \mathrm{Br}(k_v)$ vanishes for $v \notin T$ and the evaluation map $ev_A : X(k_v) \to \mathrm{Br}(k_v)$ is continuous for any place v. That the image of the adelic evaluation map $ev_A : X(\mathbf{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is annihilated by a positive integer is a consequence of Lemma 3.4. $\square$.

Lemma 3.4 *Let k be a field and let X be a variety over k. Let $A \in \mathrm{Br}(X)$. There exists a positive integer n depending only on A such that for any field extension $k \subset F$ the image of the evaluation map $ev_A : X(F) \to \mathrm{Br}(F)$ sending $M \in X(F)$ to $A(M) \in \mathrm{Br}(F)$ is annihilated by n.*

Proof. We may assume that X is irreducible and reduced. Let $k(X)$ be the function field of X. The torsion group $\mathrm{Br}(k(X))$ is the direct limit of the groups $\mathrm{Br}(U)$, where U is open in X. Thus there exists a non-empty open set $U \subset X$ such that the restriction of A to U is an element of $\mathrm{Br}(U)$ annihilated by some positive integer n. For any field extension F/k the image of $ev_A : U(F) \to \mathrm{Br}(F)$ is annihilated by n. Since $\dim(X \setminus U) < \dim(X)$, we can conclude by induction on dimension. □.

3.4 The Brauer–Manin pairing

By definition, *the Brauer–Manin pairing*

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is given by

$$ev_A\big(\{M_v\}\big) = \sum_{v \in \Omega} \mathrm{inv}_v A(M_v) \in \mathbb{Q}/\mathbb{Z}.$$

If X is proper, then $X(\mathbf{A}_k) = \prod_v X(k_v)$. In this case one can rewrite the pairing as

$$\prod_{v \in \Omega} X(k_v) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

For any subset $B \subset \mathrm{Br}(X)$, we let $X(\mathbf{A}_k)^B \subset X(\mathbf{A}_k)$ be the set of adelic points orthogonal to B with respect to the Brauer–Manin pairing. By the continuity of the evaluation map, it is a closed subset of $X(\mathbf{A}_k)$. When B is finite, Lemma 3.3 (iv) shows that the map $X(\mathbf{A}_k) \to \mathrm{Maps}(B, \mathbb{Q}/\mathbb{Z})$ factors through $\mathrm{Maps}(B, \mathbb{Z}/n)$ for some n, hence $X(\mathbf{A}_k)^B$ is closed and open in $X(\mathbf{A}_k)$. The set $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ is called *the Brauer–Manin set* of X. We abbreviate this notation by $X(\mathbf{A}_k)^{\mathrm{Br}}$.

BY the exact sequence (20) the set $X(\mathbf{A}_k)^B$ only depends on the image of B in the quotient $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

If, moreover, X is proper, then $X(\mathbf{A}_k)$ is compact by Tychonoff's theorem. When $X(\mathbf{A}_k)^{\mathrm{Br}}$ is empty, the compact set $X(\mathbf{A}_k)$ has a covering by open subsets $X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^b$, for all $b \in \mathrm{Br}(X)$. Hence there is a *finite* subset $B \subset \mathrm{Br}(X)$ such that

$$X(\mathbf{A}_k) = \bigcup_{b \in B} (X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^b),$$

and therefore $X(\mathbf{A}_k)^B = \emptyset$.

Let X be a variety over k. For any $A \in \operatorname{Br}(X)$ we have the basic commutative diagram:

$$\begin{array}{ccccc} X(k) & \hookrightarrow & X(\mathbf{A}_k) & & \\ \downarrow {\scriptstyle \mathrm{ev}_A} & & \downarrow {\scriptstyle \mathrm{ev}_A} & \searrow^{\theta_A} & \\ \operatorname{Br}(k) & \longrightarrow & \bigoplus_{v \in \Omega} \operatorname{Br}(k_v) & \xrightarrow{\mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} \end{array}$$

where the bottom line is the complex given by the class field theory exact sequence (20), and θ_A is the map that makes the diagram commutative. The set $X(\mathbf{A}_k)^{\mathrm{Br}}$ is the intersection of $\theta_A^{-1}(0)$ for all $A \in \operatorname{Br}(X)$.

Theorem 3.5 (Manin) *Let k be a number field and let X be a variety over k. The Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ contains the closure of the image of the diagonal map $X(k) \to X(\mathbf{A}_k)$.*

Proof. The inclusion $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}}$ follows immediately from the above diagram. Since the set $X(\mathbf{A}_k)^{\mathrm{Br}}$ is closed in $X(\mathbf{A}_k)$, it contains the closure of $X(k)$. □

Manin's 1970 observation is that this simple theorem accounts for most counter-examples to the Hasse principle known at the time. In these examples, the rôle of sequence (20) is played by some explicit form of the reciprocity law, mostly the quadratic reciprocity law.

It is common to use the following terminology:

If X is a variety over k such that $X(\mathbf{A}_k) \neq \emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, then one says that *there is a Brauer–Manin obstruction to the Hasse principle for X.*

If the inclusion $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)$ is not an equality, then one says that *there is a Brauer–Manin obstruction to strong approximation for X.* If X is proper, then this is a Brauer–Manin obstruction to weak approximation.

Proposition 3.6 *A morphism $f : X \to Y$ of varieties over a number field k induces a continuous map of their Brauer–Manin sets $X(\mathbf{A}_k)^{\mathrm{Br}} \to Y(\mathbf{A}_k)^{\mathrm{Br}}$.*

Proof. We have a continuous map of topological spaces $f : X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$ and a map of Brauer groups $f^* : \operatorname{Br}(Y) \to \operatorname{Br}(X)$. For a point $(P_v) \in X(\mathbf{A}_k)$ and $A \in \operatorname{Br}(Y)$ we have $(f^*A)(P_v) = A(f(P_v))$, hence f sends $X(\mathbf{A}_k)^{f^*A}$ to $Y(\mathbf{A}_k)^A$. Thus f sends $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{f^*\mathrm{Br}(Y)}$ to $Y(\mathbf{A}_k)^{\mathrm{Br}(Y)} = Y(\mathbf{A}_k)^{\mathrm{Br}}$. □

3.5 The structure of the Brauer–Manin set

When $\operatorname{Br}(X)$ is finite modulo $\operatorname{Br}_0(X)$, the Brauer–Manin set of X is an open and closed subset of $X(\mathbf{A}_k)$. More precisely, we have the following

Lemma 3.7 *Let X be a proper variety over a number field k. Assume that* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *is finite. Then there exists a finite set S of places of k such that*

$$X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{v \notin S} X(k_v)$$

for an open and closed set $Z \subset \prod_{v \in S} X(k_v)$.

Proof. There is a finite set $B \subset \mathrm{Br}(X)$ that generates $\mathrm{Br}(X)$ modulo $\mathrm{Br}_0(X)$. By Lemma 3.3 (iii) there is a finite set of places S such that $A(M_v) = 0$ for each $A \in B$ and any $M_v \in X(k_v)$, where $v \notin S$. Thus for each $A \in B$ the evaluation map $ev_A : X(\mathbf{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is the composition of the projection $X(\mathbf{A}_k) \to \prod_{v \in S} X(k_v)$ and a continuous map $\prod_{v \in S} X(k_v) \to \mathbb{Q}/\mathbb{Z}$. The resulting map $\prod_{v \in S} X(k_v) \to (\mathbb{Q}/\mathbb{Z})^B$ is continuous with finite image (Lemma 3.3 (iv)) thus its kernel Z is an open and closed subset of $\prod_{v \in S} X(k_v)$. □

Question Let X be a smooth, projective and geometrically integral variety over a number field k. Assume that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Can one choose S in Lemma 3.7 to be the union of the archimedean places of k and the places of bad reduction for X?

The following result gives sufficient conditions under which the answer is positive.

Theorem 3.8 (Colliot-Thélène–Skorobogatov) *Let k be a number field. Let S be a finite set of places of k containing the archimedean places, and let $\mathcal{O}_S$ be the ring of S-integers of k. Let $\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ be a smooth and proper $\mathcal{O}_S$-scheme with geometrically integral fibres. Let X/k be its generic fibre. Assume*

(i) $\mathrm{H}^1(X, \mathcal{O}_X) = 0$;

(ii) *the Néron–Severi group* $\mathrm{NS}\,(\overline{X})$ *has no torsion;*

(iii) *the transcendental Brauer group* $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *is a finite abelian group of order invertible in $\mathcal{O}_S$.*

Then $X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{v \notin S} X(k_v)$, where $Z \subset \prod_{v \in S} X(k_v)$ is an open and closed subset.

3.6 Examples

Iskovskikh's counter-example to the Hasse principle

Let $U = U_c$ be the variety

$$y^2 + z^2 = (c - x^2)(x^2 - c + 1) \neq 0,$$

where $c \in \mathbb{N}$ is congruent to 3 modulo 4. Using Hensel's lemma, one easily checks that U has points in all completions of $\mathbb{Q}$.

Consider the Azumaya algebra on U defined by the quaternion algebra $A = (c - x^2, -1)$. Let $X = X_c$ be a smooth compactification of U_c. One proves that the class of A comes from a class in $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. Since the evaluation map over a local field is locally constant, for any place v of $\mathbb{Q}$, finite or infinite, the image of the evaluation map

$$ev_A : X(\mathbb{Q}_v) \longrightarrow \mathrm{Br}(\mathbb{Q}_v) \subset \mathbb{Q}/\mathbb{Z}$$

coincides with the image of

$$ev_A : U(\mathbb{Q}_v) \longrightarrow \mathrm{Br}(\mathbb{Q}_v) \subset \mathbb{Q}/\mathbb{Z}.$$

Let $K = \mathbb{Q}(\sqrt{-1})$. Let v be place of $\mathbb{Q}$ and let w be a place of K over v. For $\rho_v \in \mathbb{Q}_v^*$ we have $(\rho_v, -1) = 0 \in \mathrm{Br}(\mathbb{Q}_v)$ if and only if ρ_v is a norm for the local extension $K_w/\mathbb{Q}_v$. We thus need to compute the images of the maps

$$\phi_v : U(\mathbb{Q}_v) \longrightarrow \mathbb{Q}_v^*/\mathrm{N}(K_w^*) \subset \mathbb{Z}/2$$

where ϕ_v sends $M_v = (x_v, y_v, z_v) \in U(\mathbb{Q}_v)$ to the class of $x_v^2 - c$.

If v splits in K, the target of ϕ_v is zero.

For $v = v_\infty$ we have $(\rho_v, -1) = 0 \in \mathrm{Br}(\mathbb{R})$ if and only if $\rho_v > 0$. The equation

$$y_\infty^2 + z_\infty^2 = (c - x_\infty^2)(x_\infty^2 - c + 1) \in \mathbb{R}^*$$

forces $c - x_\infty^2 > 0$, hence the image of ϕ_v is zero.

Suppose $v = p$ is a finite prime which is inert in K. We have $(\rho_p, -1) = 0$ if and only if $v(\rho_p)$ is even. If $v(x_v) < 0$, then $v(c - x_v^2)$ is even and thus $c - x_v^2$ is a norm. Suppose $v(x_v) \geq 0$. From the equality

$$(c - x_v^2) + (x_v^2 - c + 1) = 1$$

we deduce that at least one of $v(c - x_v^2)$ and $v(x_v^2 - c + 1)$ vanishes. From the equality

$$y_v^2 + z_v^2 = (c - x_v^2)(x_v^2 - c + 1) \in \mathbb{Q}_v^*,$$

we deduce that the sum of the valuations of $c - x_v^2$ and $x_v^2 - c + 1$ is even. Thus $v(c - x_v^2)$ is even, and the image of ϕ_v is zero.

For the unique ramified prime $v = 2$, an element $\rho_2 \in \mathbb{Q}_2^*$ is a sum of two squares if and only if it is the product of a power of 2 and a unit in $\mathbb{Z}_2^*$ which is congruent to 1 modulo 4. Write $x_2 = u/v$ with u and v in $\mathbb{Z}_2$, not both divisible by 2. Up to multiplication by a square, $c - x_2$ is equal to $cv^2 - u^2$, which by the hypothesis on c is congruent to $3v^2 - u^2$ modulo 4. Up to multiplication by a square, $x_2^2 - c + 1$ is equal to $u^2 - (c-1)v^2$ which by the hypothesis on c is congruent to $u^2 - 2v^2$ modulo 4. The possible values for (u^2, v^2) modulo 4 are $(0, 1), (1, 0), (1, 1)$. In the first and second cases, $3v^2 - u^2$ is congruent to 3 modulo 4 hence is not a norm for $K_2/\mathbb{Q}_2$. In the third case, $u^2 - 2v^2$ is congruent to 3 modulo 4, hence is not a norm for $K_2/\mathbb{Q}_2$. Since $\mathbb{Q}_2^*/\mathrm{N}(K_2^*) \cong \mathbb{Z}/2$, and the product $(c - x_2^2)(x_2^2 - c + 1) = y_2^2 + z_2^2$ is a norm, we conclude that $c - x_2^2$ is never a norm for the extension $K_2/\mathbb{Q}_2$. Thus the image of ϕ_2 is $1 \in \mathbb{Z}/2$.

For any $(M_v) \in X(\mathbf{A}_\mathbb{Q})$, we thus have

$$\sum_{v\in\Omega} \mathrm{inv}_v A(M_v) = 1/2,$$

hence $X(\mathbb{A}_\mathbb{Q})^A = \emptyset$ implying $X(\mathbb{Q}) = \emptyset$. It is known that in this case $\mathrm{Br}(X)/\mathrm{Br}_0(X) \cong \mathbb{Z}/2$, so the image of A generates this group.

The Reichardt–Lind counter-example to the Hasse principle

Let X be the smooth compactification of the smooth curve U over $\mathbb{Q}$ defined by

$$2y^2 = x^4 - 17 \neq 0.$$

One checks that $X(\mathbf{A}_\mathbb{Q}) \neq \emptyset$. (For the primes of good reduction this follows from the Hasse–Weil bound for the number of $\mathbb{F}_p$-points on a curve of genus 1 and the Hensel lemma.) One checks that the quaternion algebra $A = (y, 17)$ defines an element of $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. It is obvious that $A(U(\mathbb{R})) = 0$, hence by the continuity of ev_A we have $A(X(\mathbb{R})) = 0$. One then checks that $A(U(\mathbb{Q}_p)) = 0$ for any prime $p \neq 17$. By the continuity of ev_A we obtain $A(X(\mathbb{Q}_p)) = 0$. Next, ev_A sends $U(\mathbb{Q}_{17})$ to one point $1/2 \in \mathbb{Q}/\mathbb{Z}$. This implies $\mathrm{inv}_{17} A(X(\mathbb{Q}_{17})) = 1/2$. Thus

$$\sum_{v\in\Omega} \mathrm{inv}_v A(M_v) = 1/2$$

for any $(M_v) \in X(\mathbf{A}_\mathbb{Q})$. We conclude that $X(\mathbb{Q}) = \emptyset$.

4 When does the Brauer–Manin obstruction control rational points?

4.1 Known and conjectured results

Let X be a smooth, projective and geometrically integral variety over a number field k. When $X(k)$ is dense in $X(\mathbf{A}_k)$, weak approximation holds for X. In the previous section we have seen that this is impossible if the Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ is smaller than $X(\mathbf{A}_k)$. Thus a natural question is this: is $X(k)$ a dense subset of $X(\mathbf{A}_k)^{\mathrm{Br}}$? Write $X(k)^{cl}$ for the closure of $X(k)$ in $X(\mathbf{A}_k)$. If $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ we shall say that *weak approximation holds for the Brauer–Manin set of* X. Roughly speaking, one asks whether the Brauer–Manin obstruction is the only obstruction to weak approximation – and, in particular, to the Hasse principle – in the sense that weak approximation holds for those adelic points which are not obstructed by the Brauer group. In particular, one would like to produce geometric classes of varieties such that weak approximation holds for their Brauer–Manin sets. One would also like to construct examples when this is not so.

Definition 4.1 *A* **rationally connected** *variety over a field k is a smooth, projective and geometrically integral variety X such that over any algebraically closed field K containing k, any two K-points of X are connected by a rational curve, i.e. lie in the image of a morphism $\mathbb{P}^1_K \to X_K$.*

In the above definition one may simply assume that any two points are connected by a chain of rational curves, or even that two 'general' points are connected by such a chain.

(1) A rationally connected variety of dimension 1 is a smooth conic.

(2) A rationally connected variety of dimension 2 is a geometrically rational surface. By a theorem of Enriques, Manin, Iskovskikh, and Mori, any such surface is birationally equivalent to a surface of at least one of the following families:

(i) A smooth del Pezzo surface of degree d, where $1 \leq d \leq 9$.

(ii) A conic bundle over a conic (usually with degenerate fibres).

(3) Any geometrically unirational variety is rationally connected. (The converse is an open question.)

(4) By a theorem of Campana and Kollár–Miyaoka–Mori, any Fano variety (that is, a smooth projective variety with ample anticanonical bundle) is rationally connected. In particular, smooth hypersurfaces in $\mathbb{P}^n$ of degree $d \leq n$ are rationally connected.

(5) If $X \to Y$ is a dominant morphism of smooth, projective, geometrically integral varieties such that Y and the generic geometric fibre are rationally connected, then X is rationally connected.

(6) If X is a rationally connected variety over k, then $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$ and $\mathrm{Pic}(\overline{X})$ is a finitely generated free abelian group. In particular, $\mathrm{Br}(\overline{X})$ is finite and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite.

Conjecture (Colliot-Thélène) *If X is a rationally connected variety over a number field k, then $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$.*

This conjecture is birationally invariant.

Since $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite when X is rationally connected, the closed set $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)$ is open. In particular, if the conjecture holds, and $X(k) \neq \emptyset$, then weak weak approximation holds for X. This means that local points away from a given finite set of places can be approximated by a k-point: there is a finite set $S \subset \Omega$ such that for any finite set $T \subset \Omega$, $S \cap T = \emptyset$, the set $X(k)$ is dense in $\prod_{v\in T} X(k_v)$. Here are some of the consequences of conjectural weak weak approximation for rationally connected varieties.

(1) For any rationally connected variety X over a number field k with a k-point the set $X(k)$ is Zariski dense in X, hence infinite. Already in dimension 2, i.e. for geometrically rational surfaces, this is not known.

(2) Any finite group G is the Galois group of a Galois field extension of k. The case $k = \mathbb{Q}$ is the inverse Galois problem, a well known old open problem.

Known cases of Colliot-Thélène's conjecture

- *Smooth projective conic bundles over $\mathbb{P}^1_k$ with $r \leq 5$ geometric degenerate fibres.* The case $r \leq 3$ is easy: in this case the Hasse principle and weak approximation hold. For Châtelet surfaces, a particular kind of conic bundles with $r = 4$ (such as the surface in the Iskovskikh example), the conjecture was proved by Colliot-Thélène, Sansuc and Swinnerton-Dyer. The general case with $r = 4$ is due to Salberger and to Colliot-Thélène. The case $r = 5$ is due to Salberger and Skorobogatov. (This also implies the conjecture for del Pezzo surafces of degree 4 with a k-point.) Swinnerton-Dyer did some specific cases with $r = 6$. Short proofs of these results can be found in [Sko01, Ch. 7]. Recently the conjecture was proved by Browning–Matthiesen-Skorobogatov for conic bundles with any number of degenerate fibres when $k = \mathbb{Q}$ and each closed degenerate fibre is over a $\mathbb{Q}$-point. (The key ingredient is a theorem of Green–Tao–Ziegler proved by methods of additive combinatorics.) Under the same assumption about singular fibres, a recent theorem of Harpaz and Wittenberg proves that a variety fibred over $\mathbb{P}^1_{\mathbb{Q}}$ into varieties satisfying Colliot-Thélène's conjecture, with rationally connected generic fibre, satisfies this conjecture.

- *Smooth complete intersection of large dimension.* The conjecture is known for smooth complete intersections of two quadrics in $\mathbb{P}^n_k$: for $n \geq 5$, if there is a k-point, then weak approximation holds. For $n \geq 7$, the Hasse principle is known. (It is conjectured to hold for $n \geq 5$.) Smooth cubic hypersurfaces in $\mathbb{P}^n_{\mathbb{Q}}$ have rational points when $n \geq 9$ (Heath-Brown) and satisfy the Hasse principle for $n = 8$ (Hooley).

- *Homogeneous spaces of algebraic groups.* If a smooth, proper, geometrically integral variety X is birationally equivalent to a homogeneous space of a connected linear algebraic group with connected geometric stabilisers, then $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ is a theorem of Borovoi, which extends previous work of Sansuc.

Conditionally on the finiteness of the Tate–Shafarevich group, the Brauer–Manin set of an abelian variety A is the subgroup of $A(\mathbf{A}_k)$ generated by the closure of $A(k)$ (equal to the profinite completion of $A(k)$) and the neutral connected component of $A(\mathbf{A}_k)$ (the product of the neutral connected components of each $A(k_v)$. It is an open question if a similar description holds for smooth projective curves. (Known, by a result of Stoll, if the Jacobian has a map to an abelian variety of rank 0.) It was already Manin who showed that, conditionally on the finiteness of the Tate–Shafarevich group, the Brauer–Manin obstruction is the only obstruction to the Hasse principle for torsors of abelian varieties, e.g. for smooth projective curves of genus 1.

It was conjectured that $X(\mathbf{A}_k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ when X is a K3 surface.

4.2 Insufficiency of the Brauer–Manin obstruction

Let C be a smooth, projective, geometrically integral curve over $\mathbb{Q}$ such that $C(\mathbb{Q})$ consists of just one point, $C(\mathbb{Q}) = \{P\}$. (Poonen showed that such a curve exists for any number field k; moreover, Mazur and Rubin showed that C can be chosen to be an elliptic curve.) Let $\Pi \subset C(\mathbb{R})$ be an open interval containing P. Let $f : C \to \mathbb{P}^1_k$ be a surjective morphism unramified at P. Choose a coordinate function t on $\mathbb{A}^1_{\mathbb{Q}} = \mathbb{P}^1_{\mathbb{Q}} \setminus f(P)$ such that f is unramified above $t = 0$. We have $f(P) = \infty$. Take any $a > 0$ in $\mathbb{Q}$ such that a is an interior point of the interval $f(\Pi)$ and f is unramified above $t = a$.

Let p be a prime. There exists a quadratic form $Q(x_0, x_1, x_2)$ of rank 3 that represents zero in all completions of $\mathbb{Q}$ other than $\mathbb{Q}_p$ and $\mathbb{R}$, but not in $\mathbb{Q}_p$ or $\mathbb{R}$. We can assume that Q is positive definite. Choose $n \in \mathbb{Q}$ with $n > 0$ and $-nQ(1,0,0) \in \mathbb{Q}_p^{*2}$. Let $Y_1 \subset \mathbb{P}^3_{\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}}$ be given by $Q(x_0, x_1, x_2) + nt(t-a)x_3^2 = 0$, and let $Y_2 \subset \mathbb{P}^3_{\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}}$ be given by $Q(X_0, X_1, X_2) + n(1 - aT)X_3^2 = 0$. We glue Y_1 and Y_2 by identifying $T = t^{-1}$, $X_3 = tx_3$, and $X_i = x_i$ for $i = 0, 1, 2$. This produces a quadric bundle $Y \to \mathbb{P}^1_{\mathbb{Q}}$ with exactly two degenerate fibres (over $t = a$ and $t = 0$), each given by the quadratic form $Q(x_0, x_1, x_2)$ of rank 3. Define $X = Y \times_{\mathbb{P}^1_{\mathbb{Q}}} C$. This is a quadric bundle $X \to C$ with geometrically integral fibres.

For example, we can take $p = 2$ and consider Y defined by

$$x_0^2 + x_1^2 + x_2^2 + 7t(t-a)x_3^2 = 0.$$

Proposition 4.2 *In the above notation we have $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ and $X(k) = \emptyset$.*

Proof. Since $C(k) = \{P\}$ we have $X(\mathbb{Q}) \subset X_P$. The fibre X_P is the smooth quadric $Q(x_0, x_1, x_2) + nx_3^2 = 0$. This quadratic form is positive definite thus X_P has no points in $\mathbb{R}$ and so $X(\mathbb{Q}) = \emptyset$. By assumption X_P has local points in all completions of $\mathbb{Q}$ other than $\mathbb{Q}_p$ and $\mathbb{R}$. The condition $-nQ(1,0,0) \in \mathbb{Q}_p^{*2}$ implies that X_P contains $\mathbb{Q}_p$-points, so X_P has local points in all completions of $\mathbb{Q}$ but one. Choose $N_u \in X_P(\mathbb{Q}_\ell)$ for each prime $\ell \neq p$. Let $M \in \Pi$ be such that $f(M) = a$. Then the singular point of the real fibre X_M (the vertex of the quadratic cone) is a smooth real point of X. Take it as the real component of the adelic point (N_u) of X.

We claim that $(N_u) \in X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$.

The fibres of $X \to C$ are geometrically integral, and this implies that the natural map $\mathrm{Br}(C) \to \mathrm{Br}(X)$ is surjective. Thus it is enough to show that the adelic point on C such that all its non-real components are equal to P and its real component is M, is orthogonal to $\mathrm{Br}(C)$. The real point M is path-connected to P, so this adelic point is in the connected component of the diagonal image of the k-point P in $C(\mathbf{A}_k)$. By the continuity of the real evaluation map it is contained in $C(\mathbf{A}_k)^{\mathrm{Br}}$, so the proposition follows. □

As a concluding remark, this simple counter-example resists all attemps to explain it: étale Brauer–Manin obstruction is insufficient for this. This obstruction (which is stronger than the Brauer–Manin obstruction) is known to be equivalent to the so called torsor obstruction, which is also equivalent to the obstruction defined by Harpaz and Schlank using étale homotopy of Artin and Mazur. See [Po18] for more on this.

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